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Some Results on the Relative (k,n) Valiron Defects of Meromorphic Functions

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Abstract

In the paper we compare the relative (k,n) Valiron defect with the relative Nevanlinna defect of a meromorphic function where k and n are both non negative integers.

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1 Introduction, Definitions and Notations.

Let f be a non constant meromorphic function defined in the open complex plane $\mathbb{C}$. For $\alpha \in \mathbb{C} \cup \{\infty\}$, let $n(t, \alpha; f)$ denote the number of roots of $f = \alpha$ in $|z| \leq t$, the multiple roots being counted according to their multiplicities and $N(t, \alpha; f)$ is defined in the usual way in terms of $n(t, \alpha; f)$. Similarly, $\bar{n}(t, \alpha; f)$ denotes the number of distinct roots of $f = \alpha$ in $|z| \leq t$ and $\bar{N}(t, \alpha; f)$ is also defined in the usual way in terms of $\bar{n}(t, \alpha; f)$.

The Nevanlinna defect $\delta(\alpha; f)$ and the Valiron defect $\Delta(\alpha; f)$ of α are respectively defined in the following manner:

$$\delta(\alpha; f) = 1 - \limsup_{r\to\infty} \frac{N(r, \alpha; f)}{T(r, f)} = \liminf_{r\to\infty} \frac{m(r, \alpha; f)}{T(r, f)}$$

and

$$\Delta(\alpha, f) = 1 - \liminf_{r\to\infty} \frac{N(r, \alpha; f)}{T(r, f)} = \limsup_{r\to\infty} \frac{m(r, \alpha; f)}{T(r, f)}.$$

Milloux [2] introduced the concept of absolute defect of 'α' with respect to the derivative $f\,'$. Later Xiong [5] extended this definition.He introduced the term

$$\delta_R^{(k)}(\alpha; f) = 1 - \limsup_{r\to\infty} \frac{N(r, \alpha; f^{(k)})}{T(r, f)} \quad \text{for} \quad k = 1, 2, 3, ...$$

and called it the relative Nevanlinna defect of 'α' with respect to $f^{(k)}$.Xiong [5] has shown various relations between the usual defects and the relative defects. Singh [3] introduced the term relative defect for distinct zeros and poles and established various relations between it, relative defects and the usual defects. In the paper we call the following two terms

$${}_R\delta_{(n)}^{(k)}(\alpha; f) = 1 - \limsup_{r\to\infty} \frac{N\left(r, \alpha; f^{(k)}\right)}{T\left(r, f^{(n)}\right)}$$

and

$${}_R\Delta_{(n)}^{(k)}(\alpha; f) = 1 - \liminf_{r\to\infty} \frac{N(r, \alpha; f^{(k)})}{T(r, f^{(n)})}$$

respectively the relative (k, n) Nevanlinna defect and the relative (k, n) Valiron defect of 'α' with respect to $f^{(k)}$ for $k = 1, 2, 3, ...$ and $n = 0, 1, 2, 3, ...$ and prove various relations between them. For $n = 0$, the above definitions coincide with the relative Nevanlinna defect and the relative Valiron defect respectively.

The term $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise.We do not explain the standard definitions and notations of the value distribution theory and the Nevanlinna theory as those are available in [1].

The following definition is well known.

Definition 1. *The order ρ_f of a meromorphic function f is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

If $\rho_f < \infty$ then f is of finite order.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [4] *Let f be a meromorphic function of finite order such that $\sum_{a\neq\infty}\delta(a;f)=1$ and $\delta(\infty;f)=1$. Then for any non negative integer k,*

$$\lim_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)}=1.$$

Lemma 2. *Let f be a meromorphic function of finite order such that $\sum_{a\neq\infty}\delta(a;f)=1$ and $\delta(\infty;f)=1$. Then for any two non negative integers k and n,*

$$\lim_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f^{(n)})}=1.$$

We omit the proof of Lemma 2 because it can be carried out in the line of Lemma 1.

Lemma 3. *Let f be a meromorphic function of finite order with $\sum_{a\neq\infty}\delta(a;f)=\delta(\infty;f)=1$. Then for any α,*

$${}_R\delta_{(n)}^{(k)}(\alpha;f)=\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(n)}\right)}.$$

Proof. In view of Lemma 2 we get that

$${}_R\delta_{(n)}^{(k)}(\alpha;f)=1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(n)}\right)}$$

$$\begin{aligned}
&= 1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}.\lim_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f^{(n)})}\\
&= 1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\\
&= \liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\\
&= \liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(n)}\right)}.\lim_{r\to\infty}\frac{T(r,f^{(n)})}{T(r,f^{(k)})}\\
&= \liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(n)}\right)}.
\end{aligned}$$

Thus the lemma is established.

Lemma 4. *Let f be a meromorphic function of finite order such that* $\sum_{a\neq\infty}\delta(a;f)=\delta(\infty;f)=1$ *. Then for any 'α',*

$$ {}_R\Delta_{(n)}^{(k)}(\alpha;f)=\limsup_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{T(r,f^{(n)})}. $$

The proof is omitted.

3 Theorems.

In this section we present the main results of the paper.

Theorem 1. *Let f be a meromorphic function of finite order. Then for any two positive integers k and n,*

$$ {}_R\Delta_{(n)}^{(0)}(\infty,f)+_R\Delta_{(n)}^{(k)}(0,f)\geq_R\delta_{(n)}^{(0)}(0,f)+_R\delta_{(n)}^{(0)}(a,f)+_R\Delta_{(n)}^{(k)}(\infty,f), $$

where 'a' is any non zero finite complex number.

Proof. Let us consider the following identity

$$ \frac{a}{f}=1-\frac{f-a}{f^{(k)}}.\frac{f^{(k)}}{f}. $$

Since $m(r,\frac{1}{f})\leq m(r,\frac{a}{f})+O(1)$, we get from the above identity

$$ m(r,\frac{1}{f})\leq m(r,\frac{f-a}{f^{(k)}})+S(r,f). \tag{1} $$

Now by Nevanlinna's first fundamental theorem and Milloux's theorem $\{p.55, [1]\}$ it follows from (1) that

$$ m(r,\frac{1}{f})\leq T(r,\frac{f-a}{f^{(k)}})-N(r,\frac{f-a}{f^{(k)}})+S(r,f) $$

$$ \text{i.e.,} m(r,\frac{1}{f})\leq T(r,\frac{f^{(k)}}{f-a})-N(r,\frac{f-a}{f^{(k)}})+S(r,f) $$

$$ \text{i.e.,} m(r,\frac{1}{f})\leq N(r,\frac{f^{(k)}}{f-a})-N(r,\frac{f-a}{f^{(k)}})+S(r,f). \tag{2} $$

In view of $\{p.34, [1]\}$ it follows from (2) that

$$m(r, \frac{1}{f}) \leq N(r, f^{(k)}) + N(r, \frac{1}{f-a}) - N(r, f-a) - N(r, \frac{1}{f^{(k)}}) + S(r, f)$$

$$\text{i.e., } \liminf_{r\to\infty} \frac{m(r, \frac{1}{f})}{T(r, f^{(n)})}$$
$$\leq \liminf_{r\to\infty} \{\frac{N(r, f^{(k)})}{T(r, f^{(n)})} - \frac{N(r, f)}{T(r, f^{(n)})} - \frac{N(r, \frac{1}{f^{(k)}})}{T(r, f^{(n)})}\} + \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r, f^{(n)})}$$
$$\leq \liminf_{r\to\infty} \frac{N(r, f^{(k)})}{T(r, f^{(n)})} - \liminf_{r\to\infty} \frac{N(r, f)}{T(r, f^{(n)})} - \liminf_{r\to\infty} \frac{N(r, \frac{1}{f^{(k)}})}{T(r, f^{(n)})} + \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r, f^{(n)})}$$

$$\text{i.e., } {}_R\delta_{(n)}^{(0)}(0; f) \leq \{1 - {}_R\Delta_{(n)}^{(k)}(\infty; f)\} - \{1 - {}_R\Delta_{(n)}^{(0)}(\infty; f)\} - \{1 - {}_R\Delta_{(n)}^{(k)}(0; f)\} + \{1 - {}_R\delta_{(n)}^{(0)}(a; f)\}$$

$$\text{i.e., } {}_R\Delta_{(n)}^{(0)}(\infty; f) + {}_R\Delta_{(n)}^{(k)}(0; f) \geq {}_R\delta_{(n)}^{(0)}(0; f) + {}_R\delta_{(n)}^{(0)}(a; f) + {}_R\Delta_{(n)}^{(k)}(\infty; f).$$

This proves the theorem.

Remark 1. *The sign '$\geq$' in Theorem 1 can not be replaced by '$>$' only. This is evident from the following example.*

Example 1. *Let $f = expz$. Then ${}_R\Delta_{(n)}^{(0)}(\infty; f) = {}_R\Delta_{(n)}^{(k)}(0; f) = {}_R\Delta_{(n)}^{(k)}(\infty; f) = 1$ and ${}_R\delta_{(n)}^{(0)}(0; f) = {}_R\delta_{(n)}^{(0)}(\infty; f) = 1$. So ${}_R\delta_{(n)}^{(0)}(a; f) = 0$.*
Then ${}_R\Delta_{(n)}^{(0)}(\infty; f) + {}_R\Delta_{(n)}^{(k)}(0; f) = 2 = {}_R\delta_{(n)}^{(0)}(0; f) + {}_R\delta_{(n)}^{(0)}(a; f) + {}_R\Delta_{(n)}^{(k)}(\infty; f)$.

Theorem 2. *Let f be a meromorphic function of finite order such that $\sum_{a\neq\infty} \delta(a; f) = \delta(\infty; f) = 1$. Then for any two positive integers k and n,*

$${}_R\Delta_{(n)}^{(k)}(0; f) \geq \delta_{(n)}^{(0)}(0; f).$$

Proof. Since $f = f^{(k)} . \frac{f}{f^{(k)}}$ we get,

$$m(r, f) \leq m(r, f^{(k)}) + m(r, \frac{f}{f^{(k)}}). \quad (3)$$

Now by Nevanlinna's first fundamental theorem and Milloux's theorem $\{p.55, [1]\}$ we obtain from (3) that

$$m(r,f) \le m(r,f^{(k)}) + T(r,\frac{f}{f^{(k)}}) - N(r,\frac{f}{f^{(k)}})$$

$$\text{i.e.,} m(r,f) \le m(r,f^{(k)}) + T(r,\frac{f^{(k)}}{f}) - N(r,\frac{f}{f^{(k)}}) + O(1)$$

$$\text{i.e.,} m(r,f) \le m(r,f^{(k)}) + N(r,\frac{f^{(k)}}{f}) - N(r,\frac{f}{f^{(k)}}) + S(r,f). \tag{4}$$

Now in view of $\{p.34, [1]\}$ it follows from (4) that

$$\begin{aligned} m(r,f) \le\ & m(r,f^{(k)}) + N(r,f^{(k)}) + N(r,\frac{1}{f}) \\ & -N(r,f) - N(r,\frac{1}{f^{(k)}}) + S(r,f) \end{aligned}$$

$$\begin{aligned} \text{i.e.,}\liminf_{r\to\infty}\frac{m(r,f)}{T(r,f^{(n)})} \le\ & \liminf_{r\to\infty}\{\frac{N(r,f^{(k)})}{T(r,f^{(n)})} - \frac{N(r,f)}{T(r,f^{(n)})} \\ & -\frac{N(r,\frac{1}{f^{(k)}})}{T(r,f^{(n)})}\} + \limsup_{r\to\infty}\{\frac{N(r,\frac{1}{f})}{T(r,f^{(n)})} + \frac{m(r,f^{(k)})}{T(r,f^{(n)})}\} \end{aligned}$$

$$\begin{aligned} \le\ & \liminf_{r\to\infty}\frac{N(r,f^{(k)})}{T(r,f^{(n)})} - \liminf_{r\to\infty}\frac{N(r,f)}{T(r,f^{(n)})} - \liminf_{r\to\infty}\frac{N(r,\frac{1}{f^{(k)}})}{T(r,f^{(n)})} \\ & +\limsup_{r\to\infty}\frac{N(r,\frac{1}{f})}{T(r,f^{(n)})} + \limsup_{r\to\infty}\frac{m(r,f^{(k)})}{T(r,f^{(n)})}. \end{aligned} \tag{5}$$

Since $\delta_{(n)}^{(0)}(\infty;f) = \Delta_{(n)}^{(0)}(\infty;f) = 1$, by Lemma 4 we obtain from (5) that

$$\begin{aligned} \delta_{(n)}^{(0)}(\infty;f) \le\ & \{1 -_R \Delta_{(n)}^{(k)}(\infty;f)\} - \{1 - \Delta_{(n)}^{(0)}(\infty;f)\} \\ & -\{1 -_R \Delta_{(n)}^{(k)}(0;f)\} + \{1 - \delta_{(n)}^{(0)}(0;f)\} +_R \Delta_{(n)}^{(k)}(\infty;f) \end{aligned}$$

i.e., $\ {}_R\Delta_{(n)}^{(k)}(0;f) \ge \delta_{(n)}^{(0)}(0;f)$.

Thus the theorem is established.

Remark 2. *Considering* $f = expz$ *one can easily verify that '*$\ge$*' cannot be replaced by '*$>$*' only in Theorem* 2.

Theorem 3. *Let f be a meromorphic function of finite order and a, b be any two distinct finite complex numbers. Then for any two positive integers k and n,*

$$ {}_R\Delta^{(k)}_{(n)}(0; f) + \Delta^{(0)}_{(n)}(\infty; f) \geq_R \Delta^{(k)}_{(n)}(\infty; f) + \delta^{(0)}_{(n)}(a; f) + \frac{1}{2}\delta^{(0)}_{(n)}(b; f). $$

Proof. Considering the identity

$$ \frac{b-a}{f-a} = \frac{f^{(k)}}{f-a}\{\frac{f-a}{f^{(k)}} - \frac{f-b}{f^{(k)}}\} $$

we obtain in view of Milloux's theorem $\{p.55, [1]\}$,

$$ m(r, \frac{b-a}{f-a}) \leq m(r, \frac{f-a}{f^{(k)}}) + m(r, \frac{f-b}{f^{(k)}}) + S(r, f) $$

$$ \begin{aligned} \text{i.e.,} m(r, \frac{b-a}{f-a}) \;\; &\leq \;\; T(r, \frac{f-a}{f^{(k)}}) - N(r, \frac{f-a}{f^{(k)}}) \\ &+T(r, \frac{f-b}{f^{(k)}}) - N(r, \frac{f-b}{f^{(k)}}) + S(r, f). \end{aligned} \tag{6} $$

Since $m(r, \frac{1}{f-a}) \leq m(r, \frac{b-a}{f-a}) + O(1)$ and $T(r, f) = T(r, \frac{1}{f}) + O(1)$, it follows from (6) that

$$ \begin{aligned} m(r, \frac{1}{f-a}) \;\; &\leq \;\; N(r, \frac{f^{(k)}}{f-a}) - N(r, \frac{f-a}{f^{(k)}}) + N(r, \frac{f^{(k)}}{f-b}) \\ &-N(r, \frac{f-b}{f^{(k)}}) + S(r, f). \end{aligned} \tag{7} $$

In view of $\{p.34, [1]\}$ we get from (7) that

$$ \begin{aligned} m(r, \frac{1}{f-a}) \;\; &\leq \;\; N(r, f^{(k)}) + N(r, \frac{1}{f-a}) - N(r, f-a) \\ &-N(r, \frac{1}{f^{(k)}}) + N(r, f^{(k)}) + N(r, \frac{1}{f-b}) - N(r, f-b) \\ &-N(r, \frac{1}{f^{(k)}}) + S(r, f) \end{aligned} $$

$$ \begin{aligned} \text{i.e.,} \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f^{(n)})} \;\; &\leq \;\; 2\{\liminf_{r\to\infty} \frac{N(r, f^{(k)})}{T(r, f^{(n)})} - \liminf_{r\to\infty} \frac{N(r, \frac{1}{f^{(k)}})}{T(r, f^{(n)})} \\ &-\liminf_{r\to\infty} \frac{N(r, f)}{T(r, f^{(n)})}\} + \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r, f^{(n)})} \\ &+ \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-b})}{T(r, f^{(n)})} \end{aligned} $$

$$\text{i.e., } \delta^{(0)}_{(n)}(a;f) \le 2\{1 -_R \Delta^{(k)}_{(n)}(\infty;f)\} - 2\{1 -_R \Delta^{(k)}_{(n)}(0;f)\}$$
$$-2\{1 - \Delta^{(0)}_{(n)}(\infty;f)\} + \{1 - \delta^{(0)}_{(n)}(a;f)\} + \{1 - \delta^{(0)}_{(n)}(b;f)\}$$

$$\text{i.e.,} 2\delta^{(0)}_{(n)}(a;f) \le 2_R\Delta^{(k)}_{(n)}(0;f) + 2\Delta^{(0)}_{(n)}(\infty;f) - 2_R\Delta^{(k)}_{(n)}(\infty;f) - \delta^{(0)}_{(n)}(b;f)$$

$$\text{i.e.,}_R\Delta^{(k)}_{(n)}(0;f) + \Delta^{(0)}_{(n)}(\infty;f) \ge_R \Delta^{(k)}_{(n)}(\infty;f) + \delta^{(0)}_{(n)}(a;f) + \tfrac{1}{2}\delta^{(0)}_{(n)}(b;f).$$

This proves the theorem.

Theorem 4. *Let f be a meromorphic function of finite order. Then for any three positive integers n, k and p with $n > k$*

$${}_R\Delta^{(k)}_{(p)}(\infty;f) +_R \Delta^{(k)}_{(p)}(0;f) \ge_R \Delta^{(k)}_{(p)}(\infty;f) +_R \delta^{(k)}_{(p)}(a;f) + \delta^{(0)}_{(p)}(a;f),$$

where a is any finite non zero complex number.

Proof. From the identity

$$\frac{1}{f-a} = \frac{1}{a}\left\{\frac{f^{(k)}}{f-a} - \frac{f^{(k)}-a}{f^{(n)}}.\frac{f^{(n)}}{f-a}\right\}$$

and by Milloux's theorem $\{p.55, [1]\}$ we get that

$$m(r, \frac{1}{f-a}) \le m(r, \frac{f^{(k)}-a}{f^{(n)}}) + S(r,f). \tag{8}$$

Now by Nevanlinna's first fundamental theorem it follows from (8),

$$m(r, \frac{1}{f-a}) \le T(r, \frac{f^{(k)}-a}{f^{(n)}}) - N(r, \frac{f^{(k)}-a}{f^{(n)}}) + S(r,f)$$

$$\text{i.e.,} m(r, \frac{1}{f-a}) \le T(r, \frac{f^{(n)}}{f^{(k)}-a}) - N(r, \frac{f^{(k)}-a}{f^{(n)}}) + S(r,f)$$

$$\text{i.e.,} m(r, \frac{1}{f-a}) \le N(r, \frac{f^{(n)}}{f^{(k)}-a}) - N(r, \frac{f^{(k)}-a}{f^{(n)}}) + S(r,f). \tag{9}$$

Now in view of $\{p.34, [1]\}$ we obtain from (9) that

$$m(r, \frac{1}{f-a}) \le N(r, f^{(n)}) + N(r, \frac{1}{f^{(k)}-a}) - N(r, f^{(k)}-a)$$
$$-N(r, \frac{1}{f^{(n)}}) + S(r,f)$$

$$\text{i.e.,} \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f^{(p)})} \leq \liminf_{r\to\infty} \{\frac{N(r, f^{(n)})}{T(r, f^{(p)})} - \frac{N(r, f^{(k)})}{T(r, f^{(p)})} - \frac{N(r, \frac{1}{f^{(n)}})}{T(r, f^{(p)})}\} + \limsup_{r\to\infty} \{\frac{N(r, \frac{1}{f^{(k)}-a})}{T(r, f^{(p)})}\}$$

$$\text{i.e.,} \quad \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f^{(p)})} \leq \liminf_{r\to\infty} \frac{N(r, f^{(n)})}{T(r, f^{(p)})} - \liminf_{r\to\infty} \frac{N(r, f^{(k)})}{T(r, f^{(p)})} - \liminf_{r\to\infty} \frac{N(r, \frac{1}{f^{(n)}})}{T(r, f^{(p)})} + \limsup_{r\to\infty} \frac{N(r, \frac{1}{f^{(k)}-a})}{T(r, f^{(p)})}$$

$$\text{i.e.,} \quad \delta^{(0)}_{(p)}(a; f) \leq \{1 -_R \Delta^{(n)}_{(p)}(\infty; f)\} - \{1 -_R \Delta^{(k)}_{(p)}(\infty; f)\} - \{1 -_R \Delta^{(n)}_{(p)}(0; f)\} + \{1 -_R \delta^{(k)}_{(p)}(a; f)\}$$

$$\text{i.e.,} \ {}_R\Delta^{(k)}_{(p)}(\infty; f) +_R \Delta^{(n)}_{(p)}(0; f) \geq_R \Delta^{(n)}_{(p)}(\infty; f) +_R \delta^{(k)}_{(p)}(a; f) + \delta^{(0)}_{(p)}(a; f).$$

Thus the theorem is established.

Theorem 5. *Let f be a meromorphic function of finite order such that $\sum_{a\neq\infty} \delta(a; f) = \delta(\infty; f) = 1$, a be a finite complex number and b, c be two distinct non zero complex numbers. Then for any two positive integers k and n,*

$$\Delta^{(0)}_{(n)}(a; f) +_R \delta^{(k)}_{(n)}(b; f) +_R \delta^{(k)}_{(n)}(c; f) \leq 2.$$

Proof. Since $\frac{1}{f-a} = \frac{f^{(k)}}{f-a} \cdot \frac{1}{f^{(k)}}$ by Milloux's theorem $\{p.55, [1]\}$ we obtain

$$m(r, \frac{1}{f-a}) \leq m(r, \frac{1}{f^{(k)}}) + S(r, f). \tag{10}$$

Applying Nevanlinna's first fundamental theorem we get from (10) that

$$m(r, \frac{1}{f-a}) \leq T(r, f^{(k)}) - N(r, \frac{1}{f^{(k)}}) + S(r, f). \tag{11}$$

Now by Nevanlinna's second fundamental theorem and Lemma 2 it follows from (11) that

$$m(r, \frac{1}{f-a}) \leq \bar{N}(r, \frac{1}{f^{(k)}}) + \bar{N}(r, \frac{1}{f^{(k)}-b}) + \bar{N}(r, \frac{1}{f^{(k)}-c}) - N(r, \frac{1}{f^{(k)}}) + S(r, f). \tag{12}$$

Since $\bar{N}(r, \frac{1}{f^{(k)}}) - N(r, \frac{1}{f^{(k)}}) \leq 0$, we obtain from (12) that

$$m(r, \frac{1}{f-a}) \leq N(r, \frac{1}{f^{(k)}-b}) + N(r, \frac{1}{f^{(k)}-c}) + S(r, f)$$

$$\text{i.e.,} m(r, \frac{1}{f-a}) \leq T(r, \frac{1}{f^{(k)}-b}) - m(r, \frac{1}{f^{(k)}-b}) + T(r, \frac{1}{f^{(k)}-c}) - m(r, \frac{1}{f^{(k)}-c}) + S(r, f)$$

$$\text{i.e.,} m(r, \frac{1}{f-a}) \leq 2T(r, f^{(k)}) - m(r, \frac{1}{f^{(k)}-b}) - m(r, \frac{1}{f^{(k)}-c}) + S(r, f)$$

$$i.e., \limsup_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f^{(n)})} \leq 2\limsup_{r\to\infty} \frac{T(r, f^{(k)})}{T(r, f^{(n)})} - \liminf_{r\to\infty} \frac{m(r, \frac{1}{f^{(k)}-b})}{T(r, f^{(n)})} - \liminf_{r\to\infty} \frac{m(r, \frac{1}{f^{(k)}-c})}{T(r, f^{(n)})}. \quad (13)$$

Now by Lemma 2 and Lemma 3 we get from (13) that,

$$\Delta^{(0)}_{(n)}(a; f) \leq 2 -_R \delta^{(k)}_{(n)}(b; f) -_R \delta^{(k)}_{(n)}(c; f)$$

$$\text{i.e., } \Delta^{(0)}_{(n)}(a; f) +_R \delta^{(k)}_{(n)}(b; f) +_R \delta^{(k)}_{(n)}(c; f) \leq 2.$$

This proves the theorem.

Theorem 6. *Let f be a meromorphic function of finite order satisfying $\sum_{a\neq\infty} \delta(a; f) = 1$ and $\delta(\infty; f) = 1$. Then for any two positive integers k and n*

$$\delta^{(0)}_{(n)}(0, f) +_R \Delta^{(k)}_{(n)}(\alpha; f) \leq 1 \quad \textit{where } \alpha \textit{ is a non zero finite}$$

complex number.

Proof. Considering the identity

$$\frac{\alpha}{f} = \frac{f^{(k)}}{f} - \frac{f^{(k)} - \alpha}{f^{(k+1)}}.\frac{f^{(k+1)}}{f},$$

we get in view of Milloux's theorem $\{p.55, [1]\}$ and Nevanlinna's first fundamental theorem,

$$m(r, \frac{1}{f}) \leq m(r, \frac{f^{(k)} - \alpha}{f^{(k+1)}}) + S(r, f)$$

$$\text{i.e.,} m(r, \frac{1}{f}) \leq T(r, \frac{f^{(k)} - \alpha}{f^{(k+1)}}) - N(r, \frac{f^{(k)} - \alpha}{f^{(k+1)}}) + S(r, f)$$

$$\text{i.e.,} m(r, \frac{1}{f}) \leq T(r, \frac{f^{(k+1)}}{f^{(k)} - \alpha}) - N(r, \frac{f^{(k)} - \alpha}{f^{(k+1)}}) + S(r, f)$$

$$\text{i.e.,} m(r, \frac{1}{f}) \leq N(r, \frac{f^{(k+1)}}{f^{(k)} - \alpha}) - N(r, \frac{f^{(k)} - \alpha}{f^{(k+1)}}) + S(r, f). \quad (14)$$

Now in view of $\{p.34, [1]\}$ it follows from (14) that

$$\begin{aligned} m(r, \frac{1}{f}) \quad \leq \quad & N(r, f^{(k+1)}) + N(r, \frac{1}{f^{(k)} - \alpha}) - N(r, f^{(k)} - \alpha) \\ & -N(r, \frac{1}{f^{(k+1)}}) + S(r, f) \end{aligned}$$

$$\begin{aligned} \text{i.e.,} m(r, \frac{1}{f}) \quad \leq \quad & \{N(r, \frac{1}{f^{(k)} - \alpha}) - N(r, \frac{1}{f^{(k+1)}})\} + \{N(r, f^{(k+1)}) - N(r, f^{(k)})\} \\ & +S(r, f) \end{aligned}$$

i.e., $m(r, \frac{1}{f}) \leq N(r, \alpha; f^{(k)}) + \bar{N}(r, f) + S(r, f)$

Since $\delta(\infty; f) = 1$, it follows that $\lim_{r \to \infty} \frac{\bar{N}(r, f)}{T(r, f^{(n)})} = 0$.
So from above we get that

$$\liminf_{r \to \infty} \frac{m(r, 0; f)}{T(r, f^{(n)})} \leq \liminf_{r \to \infty} \frac{N(r, \alpha; f^{(k)})}{T(r, f^{(n)})}$$

$$\text{i.e., } \delta_{(n)}^{(0)}(0; f) +_R \Delta_{(n)}^{(k)}(\alpha; f) \leq 1.$$

Since $\sum_{a \neq \infty} \delta(a; f) = \delta(\infty; f) = 1$, by Lemma 4 we note that

$$_R\Delta_{(n)}^{(k)}(\alpha; f) \geq 0.$$

Thus the theorem is established.

Remark 3. *The inequality in Theorem* 6 *is best possible in the sense that '*$\leq$*' cannot be replaced by '*$<$*' only which is evident from the following example.*

Example 2. *Let* $f = expz$.

$$So\ \delta^{(0)}_{(n)}(0, f) = 1\ and\ {}_R\delta^{(k)}_{(n)}(0; f) ={}_R \delta^{(k)}_{(n)}(\infty; f) = 1.$$

Now by Nevanlinna's second fundamental theorem and in view of above we get that

$$\begin{aligned} T(r, f^{(k)}) &\leq N(r, a; f^{(k)}) + S(r, f^{(k)}) \\ &\leq T(r, f^{(k)}) + S(r, f^{(k)}) \end{aligned}$$

$$\begin{aligned} i.e., \frac{T(r, f^{(k)})}{T(r, f^{(n)})} &\leq \frac{N(r, a; f^{(k)})}{T(r, f^{(n)})} + \frac{S(r, f^{(k)})}{T(r, f^{(k)})} . \frac{T(r, f^{(k)})}{T(r, f^{(n)})} \\ i.e., \frac{T(r, f^{(k)})}{T(r, f^{(n)})} &\leq \frac{T(r, f^{(k)})}{T(r, f^{(n)})} + \frac{S(r, f^{(k)})}{T(r, f^{(k)})} . \frac{T(r, f^{(k)})}{T(r, f^{(n)})} . \end{aligned}$$

By Lemma 2 *it follows from above that*

$$\lim_{r \to \infty} \frac{N(r, a; f^{(k)})}{T(r, f^{(n)})} = 1.$$

Therefore

$${}_R\Delta^{(k)}_{(n)}(\alpha; f) = 0.$$

Thus

$$\delta^{(0)}_{(n)}(0; f) +_R \Delta^{(k)}_{(n)}(\alpha; f) = 1.$$

Theorem 7. *Let* k *and* n *be any two positive integers and* a *be a finite complex number. Then for any meromorphic function* f *of finite order satisfying* $\sum_{a \neq \infty} \delta(a; f) = \delta(\infty; f) = 1$,

$${}_R\Delta^{(k)}_{(n)}(a; f) \geq_R \delta^{(k)}_{(n)}(a; f) + \delta^{(0)}_{(n)}(0; f).$$

Proof. Let $b \neq a$ be a finite complex number.
Since

$$\frac{a - b}{f^{(k)} - a} = \frac{f}{f^{(k)} - a} \{ \frac{f^{(k)} - b}{f} - \frac{f^{(k)} - a}{f} \},$$

we obtain in view of Milloux's theorem $\{p.55, [1]\}$ and Nevanlinna's first fundamental theorem,

$$m(r, \frac{a-b}{f^{(k)}-a}) \leq m(r, \frac{f}{f^{(k)}-a}) + S(r, f)$$

$$\text{i.e.,} m(r, \frac{1}{f^{(k)}-a}) \leq T(r, \frac{f}{f^{(k)}-a}) - N(r, \frac{f}{f^{(k)}-a}) + S(r, f)$$

$$\text{i.e.,} m(r, \frac{1}{f^{(k)}-a}) \leq T(r, \frac{f^{(k)}-a}{f}) - N(r, \frac{f}{f^{(k)}-a}) + S(r, f)$$

$$\text{i.e.,} m(r, \frac{1}{f^{(k)}-a}) \leq N(r, \frac{f^{(k)}-a}{f}) - N(r, \frac{f}{f^{(k)}-a}) + S(r, f). \tag{15}$$

In view of $\{p.34, [1]\}$ it follows from (15) that

$$\begin{aligned} m(r, \frac{1}{f^{(k)}-a}) \;\; \leq \;\; & N(r, f^{(k)}-a) + N(r, \frac{1}{f}) - N(r, f) \\ & -N(r, \frac{1}{f^{(k)}-a}) + S(r, f) \end{aligned}$$

$$\begin{aligned} \text{i.e.,} \liminf_{r\to\infty} \frac{m(r, \frac{1}{f^{(k)}-a})}{T(r, f^{(n)})} \;\; \leq \;\; & \liminf_{r\to\infty} \{ \frac{N(r, f^{(k)})}{T(r, f^{(n)})} - \frac{N(r, f)}{T(r, f^{(n)})} - \frac{N(r, \frac{1}{f^{(k)}-a})}{T(r, f^{(n)})} \} \\ & + \limsup_{r\to\infty} \frac{N(r, \frac{1}{f})}{T(r, f^{(n)})} \end{aligned}$$

$$\begin{aligned} \text{i.e.,} \liminf_{r\to\infty} \frac{m(r, \frac{1}{f^{(k)}-a})}{T(r, f^{(n)})} \;\; \leq \;\; & \liminf_{r\to\infty} \frac{N(r, f^{(k)})}{T(r, f^{(n)})} - \liminf_{r\to\infty} \frac{N(r, f)}{T(r, f^{(n)})} \\ & - \liminf_{r\to\infty} \frac{N(r, \frac{1}{f^{(k)}-a})}{T(r, f^{(n)})} + \limsup_{r\to\infty} \frac{N(r, \frac{1}{f})}{T(r, f^{(n)})}. \end{aligned} \tag{16}$$

Since $\delta(\infty; f) = 1$,

$$\lim_{r\to\infty} \frac{N(r, f)}{T(r, f^{(n)})} = \lim_{r\to\infty} \frac{\bar{N}(r, f)}{T(r, f^{(n)})} = 0$$

and so

$$\lim_{r\to\infty} \frac{N(r, f^{(k)})}{T(r, f^{(n)})} = \lim_{r\to\infty} \frac{N(r, f)}{T(r, f^{(n)})} + k \lim_{r\to\infty} \frac{\bar{N}(r, f)}{T(r, f^{(n)})} = 0.$$

Thus by Lemma 3 it follows from (16) that

$$ {}_R\delta^{(k)}_{(n)}(a;f) \leq_R \Delta^{(k)}_{(n)}(a;f) - 1 + \{1 - \delta^{(0)}_{(n)}(0;f)\} $$

$$ \text{i.e., } {}_R\Delta^{(k)}_{(n)}(a;f) \geq_R \delta^{(k)}_{(n)}(a;f) + \delta^{(0)}_{(n)}(0;f). $$

This proves the theorem.

Theorem 8. *Let f be a meromorphic function of finite order such that $\sum\limits_{\alpha\neq\infty}\delta(\alpha;f) = \delta(\infty;f) = 1$ and $a_1, a_2, ...a_q$ are all distinct finite complex numbers. Then for any three positive integers n, k and p with $k > n$,*

$$ \sum_{i=1}^{q} {}_R\Delta^{(k)}_{(p)}(a_i;f) +_R \Delta^{(n)}_{(p)}(0;f) \geq \sum_{i=1}^{q} {}_R\delta^{(k)}_{(p)}(a_i;f) + q_R\delta^{(n)}_{(p)}(0;f). $$

Proof. Let $F = \sum\limits_{i=1}^{q} \frac{1}{f^{(k)} - a_i}$ for $i = 1, 2, ...q$.
Then we get

$$ \sum_{i=1}^{q} m(r, \frac{1}{f^{(k)} - a_i}) \leq m(r, F) + O(1) $$

$$ i.e., \sum_{i=1}^{q} m(r, \frac{1}{f^{(k)} - a_i}) \leq m(r, \sum_{i=1}^{q} \frac{f^{(n)}}{f^{(k)} - a_i}) + m(r, \frac{1}{f^{(n)}}) + O(1) $$

$$ i.e., \sum_{i=1}^{q} m(r, \frac{1}{f^{(k)} - a_i}) \leq m(r, \frac{1}{f^{(n)}}) + \sum_{i=1}^{q} m(r, \frac{f^{(n)}}{f^{(k)} - a_i}) + O(1) $$

$$ i.e., \sum_{i=1}^{q} m(r, \frac{1}{f^{(k)} - a_i}) \leq m(r, \frac{1}{f^{(n)}}) + \sum_{i=1}^{q} \{T(r, \frac{f^{(n)}}{f^{(k)} - a_i}) - N(r, \frac{f^{(n)}}{f^{(k)} - a_i})\} + O(1) $$

$$ i.e., \sum_{i=1}^{q} m(r, \frac{1}{f^{(k)} - a_i}) \leq m(r, \frac{1}{f^{(n)}}) + \sum_{i=1}^{q} \{T(r, \frac{f^{(k)} - a_i}{f^{(n)}}) - N(r, \frac{f^{(n)}}{f^{(k)} - a_i})\} + O(1) $$

$$\text{i.e.,} \sum_{i=1}^{q} m(r, \frac{1}{f^{(k)} - a_i}) \leq m(r, \frac{1}{f^{(n)}}) + \sum_{i=1}^{q} \{N(r, \frac{f^{(k)} - a_i}{f^{(n)}}) - N(r, \frac{f^{(n)}}{f^{(k)} - a_i})\} + S(r, f). \tag{17}$$

In view of $\{p.34, [1]\}$ we get from(17)that

$$\sum_{i=1}^{q} m(r, \frac{1}{f^{(k)} - a_i}) \leq m(r, \frac{1}{f^{(n)}}) + \sum_{i=1}^{q} \{N(r, f^{(k)} - a_i) + N(r, \frac{1}{f^{(n)}}) - N(r, f^{(n)}) - N(r, \frac{1}{f^{(k)} - a_i})\} + S(r, f)$$

$$\text{i.e.,} \sum_{i=1}^{q} m(r, \frac{1}{f^{(k)} - a_i}) \leq m(r, \frac{1}{f^{(n)}}) + qN(r, f^{(k)}) + qN(r, \frac{1}{f^{(n)}}) - qN(r, f^{(n)}) - \sum_{i=1}^{q} N(r, \frac{1}{f^{(k)} - a_i}) + S(r, f)$$

$$\text{i.e.,} \liminf_{r\to\infty} \frac{\sum_{i=1}^{q} m(r, \frac{1}{f^{(k)} - a_i})}{T(r, f^{(p)})} \leq \liminf_{r\to\infty} \{\frac{qN(r, f^{(k)})}{T(r, f^{(p)})} - \frac{qN(r, f^{(n)})}{T(r, f^{(p)})} - \frac{\sum_{i=1}^{q} N(r, \frac{1}{f^{(k)} - a_i})}{T(r, f^{(p)})}\} + \limsup_{r\to\infty} \{\frac{m(r, \frac{1}{f^{(n)}})}{T(r, f^{(p)})} + \frac{qN(r, \frac{1}{f^{(n)}})}{T(r, f^{(p)})}\}$$

$$\text{i.e.,} \sum_{i=1}^{q} \liminf_{r\to\infty} \frac{m(r, \frac{1}{f^{(k)} - a_i})}{T(r, f^{(p)})} \leq q \liminf_{r\to\infty} \frac{N(r, f^{(k)})}{T(r, f^{(p)})} - q \liminf_{r\to\infty} \frac{N(r, f^{(n)})}{T(r, f^{(p)})} - \sum_{i=1}^{q} \liminf_{r\to\infty} \frac{N(r, \frac{1}{f^{(k)} - a_i})}{T(r, f^{(p)})} + \limsup_{r\to\infty} \frac{m(r, \frac{1}{f^{(n)}})}{T(r, f^{(p)})} + q \limsup_{r\to\infty} \frac{N(r, \frac{1}{f^{(n)}})}{T(r, f^{(p)})}. \tag{18}$$

Since $\delta(\infty; f) = 1$,

$$\lim_{r\to\infty} \frac{N(r, f)}{T(r, f)} = \lim_{r\to\infty} \frac{\bar{N}(r, f)}{T(r, f)} = 0.$$

So

$$\lim_{r\to\infty}\frac{N(r,f^{(k)})}{T(r,f^{(p)})}=\lim_{r\to\infty}\frac{N(r,f)}{T(r,f^{(p)})}+k\lim_{r\to\infty}\frac{\bar{N}(r,f)}{T(r,f^{(p)})}=0.$$

Similarly

$$\lim_{r\to\infty}\frac{N(r,f^{(n)})}{T(r,f^{(p)})}=0.$$

Now by Lemma 3 and Lemma 4 it follows from (18) that

$$\sum_{i=1}^{q}\ {}_R\delta_{(p)}^{(k)}(a_i;f)\leq\sum_{i=1}^{q}\ {}_R\Delta_{(p)}^{(k)}(a_i;f)-q+_R\Delta_{(p)}^{(n)}(0;f)+q\{1-_R\delta_{(p)}^{(n)}(0;f)\}$$

i.e., $\sum_{i=1}^{q}\ {}_R\Delta_{(p)}^{(k)}(a_i;f)+_R\Delta_{(p)}^{(n)}(0;f)\geq\sum_{i=1}^{q}\ {}_R\delta_{(p)}^{(k)}(a_i;f)+q_R\delta_{(p)}^{(n)}(0;f)$.

Thus the theorem is established.

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