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On the Diophantine Equation $x^2 + 11^{2k+1} = y^n$

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Abstract. In this paper, it has been proved that if n is an odd integer > 3 and $h = 1$ is the class number of the field $Q\left(\sqrt{-11}\right)$, so the ring of integers $Q\left(\sqrt{-11}\right)$ is unique factorization domain and then the diophantine equation $x^2 + 11^{2k+1} = y^n$ has exactly one family of solution (n, k, x, y).

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1. INTRODUCTION

Many cases of the Diophantine Equation $x^2 + q^{2k+1} = y^n$, where q is a prime , k, x, n, y are positive integers have been studied in last few years.Cohn [3] has given a different proof from Nagell's proof for $k = 0$. The case $q = 3, 5$ was studied by S.A.Arif and Fadwa S. Abu Muriefah $[1], [2]$. R. Alter and K.Kubato consider the diophantine equation $x^2 + 11 = 3^n$ and gave a related sequence in his paper.In this paper, we study the equation $x^2 + 11^{2k+1} = y^n$ and we prove with different method the following theorem.For proof of the theorem firstly we must give a lemma.

Lemma 1. *The equation* $11x^2+1=y^n$, *where* n *is an odd integer* ≥ 3 *has no solution in integers* x *and* y, *for* y *odd* [6].

Theorem 1. *The equation*

$$x^2+11^{2k+1}=y^n \qquad (1)$$

when n *is an odd integer* ≥ 3, *and* $k\geq 0$, $h=1$ *is the class number of the field* $\mathbb{Q}\left(\sqrt{-11}\right)$, *has exactly only one family of solution given by*

$$n=3, \qquad k=3M, \qquad x=4.11^{3M}, \qquad y=3.11^{2M}$$

Proof. We first consider the case when $(11,x)=1$. Let n be odd then there is no loss of generality in considering only $n=p$ an odd prime.Thus $x^2+11^{2k+1}=y^p$. Then from [3] we have only two possibilities and they are

$$x+11^k\sqrt{-11}=\left(a+b\sqrt{-11}\right)^p \qquad (2)$$

where $y=a^2+11b^2$ and

$$x+11^k\sqrt{-11}=\left(\frac{a+b\sqrt{-11}}{2}\right)^3 \qquad a\equiv b\equiv 1\,(\mathrm{mod}\,2) \qquad (3)$$

where $y=\frac{a^2+11b^2}{4}$ for some rational integers a and b.

In (2) since $y=a^2+11b^2$ and y is odd so only one of a or b is odd and the other is even . Equating imaginary parts we get

$$11^k=b\sum_{r=0}^{\frac{p-1}{2}}\left(\begin{array}{c}p\\2r+1\end{array}\right)a^{p-2r-1}\left(-11b^2\right)^r$$

so b is odd and a is even.Since 11 does not divide the term inside $\sum$ we get $b=\pm 11^k$. Hence

$$\pm 1=\sum_{r=0}^{\frac{p-1}{2}}\left(\begin{array}{c}p\\2r+1\end{array}\right)a^{p-2r-1}\left(-11b^2\right)^r$$

This is equation (1) in [3] and we can use lemma 4 and lemma 5 of [3] to show that both signs are impossible.Hence (2) gives rise to no solutions.

Now consider the equation (3). By equating imaginary parts we obtain

$$8.11^k=b.\left(3a^2-11b^2\right) \qquad (4).$$

If $b=\pm 1$ in (4) we get

$$\pm 8.11^{k} = \left(3a^{2} - 11\right)$$

The case $k = 0$ then

$$\begin{aligned} \pm 8 &= 3a^{2} - 1 \\ 3 &= 3a^{2} \\ 1 &= a^{2} \\ a &= \pm 1 \end{aligned} \qquad (5)$$

$$\begin{aligned} y &= \frac{a^{2} + 11b^{2}}{4} = \frac{1 + 11.1}{4} = 3 \\ x &= \left|\frac{a^{3} - 3.11.a.b^{2\cdot}8}{8}\right| = \left|\frac{-1 + 33}{8}\right| = 4. \end{aligned}$$

The case $k > 0$ then

$$\begin{aligned} \pm 8.11^{k} &= 3a^{2} - 11 \\ \pm 8.11^{k} + 11 &= 3a^{2} \\ 11\left(\pm 8.11^{k-1} + 1\right) &= 3a^{2} \end{aligned}$$

and this equation clearly has no solution.

If $b = \pm 11^{\lambda}$, $0 < \lambda < k$, then (4) becomes

$$\begin{aligned} 8.11^{\lambda} &= \pm 11^{\lambda}\left(3a^{2} - 11.11^{2\lambda}\right) \\ \pm 8.11^{k-\lambda} &= \left(3a^{2} - 11^{2\lambda+1}\right) \end{aligned}$$

this is not possible modulo 11, if $k - \lambda > 0$. So $k - \lambda = 0$ that is $k = \lambda$ then we get;

$$\pm 8 = 3a^{2} - 11^{2k+1} \qquad (6)$$

So we get the same solution $a = \pm 1$, which we find in equation (5).

And last one if $b = \pm 11^{k}$, then we have

$$\begin{aligned} 8.11^{k} &= \pm 11^{k}\left(3a^{2} - 11.11^{2k}\right) \\ \pm 8 &= \left(3a^{2} - 11^{2k+1}\right). \end{aligned}$$

So we get equation 6 , and the same solution too.

And now we suppose that $11 \mid x$ and let $x = 11^{s}X$, $y = 11^{t}Y$, where $(11, X) = (11, Y) = 1$, $s > 0$ and $t > 0$, then equation (1) becomes

$$11^{2s}X^2 + 11^{2k+1} = 11^{nt}Y^n.$$

We have three possibilities:

1. $2s = \min(2s,\ 2k+1,\ nt)$. Then we get

$$X^2 + 11^{2(k-s)+1} = 11^{nt-2s}Y^n$$

and considering this equation modulo 11. We deduce that $nt - 2s = 0$ than $x^2 + 11^{2(k-s)+1} = Y^n$ with $(11, X) = 1$. If $k - s = 0$, then this equation has the only solution $x = 4$, $n = 3$ so $nt = 3t = 2s$ that $3 \mid s$, let $s = 3M$ then we know $s = k$ i.e. $k = 3M$ and $nt = 2s \Longrightarrow nt = 2.3M$ and $t \mid 2$. Let $t = 2u$, then $n.2u = 2.3M$ i.e. $n = 3M$ and $m = 6M + 1$. Hence the solution is given by $x = 4.11^{3M}$, $y = 3.11^{2M}$.

2. $2k + 1 = \min(2s,\ 2k+1,\ nt)$. Then we get

$$11^{2s-2k-1}X^2 + 1 = 11^{nt-2k-1}Y^n$$

and considering this equation modulo 11, we get $nt - 2k - 1 = 0$ so n is odd and $11\left(11^{s-k-1}X\right)^2 + 1 = Y^n$ by the lemma 1, this equation has no solution.

3. $nt = \min(2s,\ 2k+1,\ nt)$. Then we get

$$11^{2s-nt}X^2 + 11^{2k+1-nt} = Y^n$$

and this is possible modulo 11 only if $2s - nt = 0$ or $2k + 1 - nt = 0$ and both of these cases have already been discussed.This concludes the proof. ∎

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