International Mathematical Forum, 4, 2009, no. 6, 287 - 291

Some Remarks on Multiplication and Comultiplication Modules

H. Ansari-Toroghy [1], F. Farshadifar and M. Mast-Zohouri

Department of Mathematics, Faculty of Science
Guilan University, P. O. Box 1914, Rasht, Iran

Abstract

This paper deals with some results concerning multiplication and comultiplication modules over a commutative ring.

Mathematics Subject Classification: 13C99

Keywords: multiplication and comultiplication modules

1 Introduction

Throughout this paper R will denote a commutative ring with identity. Also for an R-module M, the notation $grade(I, M)$ will denote the grade I relative to M, where R is a commutative Noetherian ring and I is an ideal of R. We will follow the terminology concerning $grade(I, M)$ and *Cohen-Macaulay modules* from [4].

An R-module M is called a *multiplication module* if for every submodule N of M there exists an ideal I od R such that $N = IM$.

An ideal I of R is said to be *second* (see [7]) if for each $r \in R$, we have $rI = 0$ or $rI = I$.

A submodule N of an R-module M is said to be *completely irreducible* (see [6]) if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M, then $N = N_i$ for some $i \in I$.

A submodule N of an R-module M is said to be *large* (see [1]) if for every non-zero submodule L of M, $N \cap L \neq 0$.

[1]ansari@guilan.ac.ir

An R-module M is said to be *cocyclic* (see [6]) if $Soc(M)$ is large and a simple submodule of M.

The sets $Ass_R(M)$ and $Supp_R(M)$ are defined as

$$Ass_R(M) = \{P \in Spec(R) : P = (0 :_R x), for\ some\ non-zero\ element\ x\ of M\};$$

$$Supp_R(M) = \{P \in Spec(R) : P \supseteq (0 :_R x), for\ some\ non-zero\ element\ x\ of M\}.$$

In [2], the dual notion of multiplication modules was introduced and the first properties of this class of modules have been considered. We recall that M is a comultiplication module (see [2]) if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$. Also it is shown that (see [2, 3.7]) M is a comultiplication module if and only if for each submodule N of M, $N = (0 :_M Ann_R(N))$.

2 Main results

Remark 2.1 (see [3]). Let M be a comultiplication R-module. Then

(a) If P is a maximal ideal of R and $(0 :_M P) \neq 0$, then $(0 :_M P)$ is simple.

(b) If B is an ideal of R such that $(0 :_M B) = 0$, then for every element $m \in M$, there exists an element b of B such that $m = bm$.

Theorem 2.2. Let M be a faithful multiplication R-module. Then we have the following.

(a) If R is a Noetherian ring and I is an ideal of R, then $grade(I, M) = grade(I, R)$.

(b) If R is a Noetherian ring, then $Ass_R(M) = Ass_R(R)$.

(c) If M is a finitely generated semisimple module, then R is a semisimple ring.

Proof. (a) First note that since R is Noetherian ring, M is a finitely generated module by [5]. It is enough to show that every sequence in I is an R-sequence if and only if it is an M-sequence. To see this, let I be an ideal of R and let $X := x_1, x_2, ..., x_n$ be an R-sequence in I. By [5, 3.1], $XR \neq R$ if and only if $XM \neq M$. Now assume that $x_i m \in (x_1, ..., x_{i-1})M$, where $1 \leq i \leq n$ and $m \in M$. Since M is a multiplication R-module, there exists an ideal J of R such that $Rm = JM$. Thus $x_i JM \subseteq (x_1, ..., x_{i-1})M$. Hence $x_i J \subseteq (x_1, ..., x_{i-1})$ by

[5, 3.1]. Therefore, $J \subseteq (x_1, ..., x_{i-1})$. So $Rm = JM \subseteq (x_1, ..., x_{i-1})M$. This implies that $m \in (x_1, ..., x_{i-1})M$. It follows that X is an M-sequence. The reverse implication is proved similarly.

(b) Let $P \in Ass_R(M)$. Then $P = (0 :_R m)$ for some $m \in M$. Since M is a multiplication R-module, there exists an ideal I of R such that $Rm = IM$. Thus $P = (0 :_R IM)$. Since M is faithful, $P = (0 :_R I)$. Since R is Noetherian there exists $a \in I$ such that $P = (0 :_R a)$. Thus $Ass_R(M) \subseteq Ass_R(R)$. Conversely, assume that $P \in Ass_R(R)$. Thus $P = (0 :_R a)$ for some $a \in R$. Since M is faithful, $P = (0 :_R aM)$. But by [5], M is Noetherian. Thus aM is finitely generated. Hence $P = (0 :_R x)$ for some $x \in aM$. Therefore $Ass_R(R) \subseteq Ass_R(M)$ as desired.

(c) Let M be a semisimple R-module and let I be a proper ideal of R. Then there exists a proper submodule N of M such that $M = N + IM$ (*d.s.*). Since M is a multiplication R-module, $N = JM$ for some ideal J of R. Thus $R = I + J$ by [5, 3.1]. By [5, 1.6], $IM \cap JM = (I \cap J)M$. Thus $(I \cap J)M = 0$. Now since M is faithful, $I \cap J = 0$ as desired.

Corollary 2.3. Let R ba a Noetherian ring and let M be a faithful multiplication R-module. Then M is a Cohen Macaulay R-module if and only if R is a Cohen-Macaulay ring.

Theorem 2.4. Let U be a comultiplication R-module. Then

(a) If N is a finitely cogenerated submodule of U, then there exists a finitely generated ideal I of R such that $N = (0 :_U I)$.

(b) $\sum_{f \in M^*} Imf = (0 :_U Ann_R(M^*))$, where $M^* = Hom_R(M, U)$.

(c) $Max(R) \cap A(U) \subseteq Ass_R(U)$, where

$$A(U) = \{P \in Spec(R) : (0 :_U P) \neq 0\}.$$

(d) $Supp_R(U) \subseteq A(U)$.

Proof. (a) Let L be a completely irreducible submodule of U. Then $L = (0 :_U I) = \cap_{a \in I}(0 :_U a)$, where $I = Ann_R(L)$. Thus $L = (0 :_U a)$ for some $a \in I$. Now since N is finitely cogenerated, $N = \cap_{i=1}^n L_i$, where L_i is a completely irreducible submodule of U for each i. Therefore, $N = \cap_{i=1}^n (0 :_U a_i)$ for some $a \in Ann_R(L_i)$. Thus $N = (0 :_U I)$, where $I = Ra_1 + Ra_2 + ... + Ra_n$.

(b) Let $V = \sum_{f \in M^*} Imf$. Then V is a submodule of U, and hence $V = (0 :_U I)$ for some ideal I of R. Let $\theta \in M^*$. Then $\theta(M) \subseteq V$. This implies that

$I\theta(M) = 0$. Thus $(I\theta)M = 0$. It follows that $I\theta = 0$. Hence $I \subseteq Ann_R(M^*)$ and $(0 :_U Ann_R(M^*)) \subseteq V$. On the other hand, for any $\phi \in M^*$,

$$\phi(Ann_R(M^*)M) = Ann_R(M^*)\phi(M) = (Ann_R(M^*)\phi)M = 0$$

Thus $Ann_R(M^*)\phi(M) = 0$. It follows that $V \subseteq (0 :_U Ann_R(M^*))$ as desired.

(c) Suppose that $P \in Max(R) \cap A(U)$. Then $(0 :_U P) \neq 0$ and it is a minimal submodule of U by Remark 2.1 (a). Hence there exists $0 \neq m \in U$ such that $(0 :_U P) = Rm$ so that $P \subseteq Ann_R(Rm)$. Since P is maximal and $0 \neq m$, $P = Ann_R(Rm)$ as desired.

(d) Suppose that $P \in Supp(U)$. Then there exists $0 \neq m \in U$ such that $(0 :_R m) \subseteq P$. Assume that $(0 :_U P) = 0$. Then by Remark 2.1 (b), there exists $p \in P$ such that $(1 - p)m = 0$. Hence $(1 - p) \in (0 :_R m) \subseteq P$, a contradiction. Therefore, $(0 :_U P) \neq 0$ as desired.

Theorem 2.5.

(a) M be a non-zero multiplication R-module and let S be a second ideal of R such that $SM = M$. Then M is a cocyclic R-module.

(b) Let R be a ring which is not a filed and let S_1 and S_2 be simple R-modules such that $S_1 + S_2$ is faithful. Then $S_1 + S_2$ is a comultiplication R-modules.

(c) Let R be a Noetherian ring and let M be a faithful divisible multiplication R-module. Then R is a semi-local ring.

Proof. (a) By [6], M has a proper completely irreducible submodule L. Since M is a multiplication R-module, $L = IM$ for some ideal I of R. Thus $L = IM = SIM$. Since S is second, $SI = 0$ or $SI = S$. Hence $L = 0$ or $L = M$. Since L is proper, $L = 0$. Therefore, M is a cocyclic R-module.

(b) Let $M = S_1 + S_2$. It is clear that $S_1 \subseteq (0 :_M Ann_R(S_1))$. Suppose that $m \in (0 :_M Ann_R(S_1))$. Then $m = m_1 + m_2$ where $m_1 \in S_1$ and $m_2 \in S_2$ and $mAnn_R(S_1) = 0$. If $m \notin S_1$, then $m_2 \notin S_1$. Since $m_2Ann_R(S_1) = 0$, we have $m_2 \in S_2 \cap (0 :_M Ann_R(S_1))$. This in turn implies that $m_2 = 0$ or $S_2 \subseteq (0 :_M Ann_R(S_1))$. Hence $M = S_1 + S_2 \subseteq (0 :_M Ann_R(S_1))$. Thus $Ann_R(S_1) = 0$. But this is a contradiction, because R is not a filed. Therefore, $m \in S_1$ as desired.

(c) Let m be a maximal ideal of R. Since R is Noetherian, M is finitely generated by [5]. Thus by [5, 3.1], $mM \neq M$. Now since M is divisible, $m \in Zd(R) = \cup_{P \in Ass(R)} P$. Thus $m \in Ass(R)$. This implies that R has a finite number of maximal ideals and the proof is completed.

References

[1] W. Anderson and K. R. Fuller, Rings and categories of modules, Springer-Verlag, New York-Heidelberg-Berlin, 1974.

[2] H. Ansari-Toroghy and F. Farshadifar, The dual notion of multiplication modules, Taiwanese Journal of Mathematics, (4) **11** (2007), 1189-1201.

[3] H. Ansari-Toroghy and F. Farshadifar, On comultiplication modules, submitted.

[4] W. Bruns, and J. Herzong, Cohen-Macaulay rings, Cambridge studies in Advanced Mathematics 39, 1996.

[5] Z.A. El-Bast and P.F. Smith, Multiplication modules, Comm. Algebra (4) **16** (1988), 755-779.

[6] L. Fuchs, W. Heinzer and B. Olberding, Commutative ideal theory without finiteness conditions: Irreducibility in the quotient filed, in preparation.

[7] S. Yassemi, The dual notion of prime submodules, Arch. Math (Brno), **37** (2001), 273-278.

Received: March 16, 2008