International Mathematical Forum, 4, 2009, no. 42, 2065 - 2074

Orthocryptic rpp Semigroups[1]

Yanping Yang and Xiaojiang Guo

Department of Mathematics
Jiangxi Normal University, Nanchang
Jiangxi 330022, P.R. China
yangyanping321@126.com
xjguo@jxnu.edu.cn

Abstract

A strongly rpp semigroup S is called *orthocryptic rpp semigroup* if (1) the set of idempotents forms a band; and (2) the relation $\mathcal{H}^{(\dagger)}$ is a congruence. Some characterizations of orthodox cryptic rpp semigroups are obtained. In particular, it is proved that a semigroup is a orthocryptic rpp semigroup if and only if it is isomorphic to the spined product of a band and a C-rpp semigroup.

Mathematics Subject Classification: 20M10

Keywords: Orthocryptic rpp semigroup; C-rpp semigroup; Cryptic rpp semigroup

1 Introduction

Throughout this paper, we follow the notations and terminologies given by Fountain [2].

Similar to rpp rings, a semigroup S is called an *rpp semigroup* if for all $a \in S$, aS^1, regarded as a right S^1-system, is projective. To study the structure of rpp semigroups, Fountain [2] considered a Green-like left congruence relation $\mathcal{L}^*$ on a semigroup S defined by: for $a, b \in S$, $a\mathcal{L}^*b$ if and only if $ax = ay \Leftrightarrow bx = by$ for all $x, y \in S^1$. Dually, we can define the right relation $\mathcal{R}^*$. It is not difficult to see that when a, b are regular elements, $a\mathcal{L}^*b$ if and only if $a\mathcal{L}b$. In the same reference [2], Fountain pointed out that a semigroup S is an rpp semigroup if and only if each $\mathcal{L}^*$-class of S contains at least one idempotent.

[1]This research is jointly supported by the Natural Science Foundation of Jiangxi Province; the Science Foundation of the Education Department of Jiangxi Province and the Foundation of Jiangxi Normal University, China.

We call a strongly rpp semigroup S cryptic rpp semigroup if $\mathcal{H}^\diamond$ is a congruence on S.

We call an rpp semigroup S a *C-rpp semigroup* if the idempotents of S are central. It is well known that a semigroup S is a C-rpp semigroup if and only if S is a strong semilattice of left cancellative monoids (see, [1]). Obviously, C-rpp semigroups are proper generalizations of Clifford semigroups. Guo-Guo-Shum [7] gave another kind of generalizations of Clifford semigroups in the range of rpp semigroups.

As an analogue of completely regular semigroups in the range of rpp semigroups, Guo-Shum-Zhu [19] defined the concept of "strongly rpp semigroup". So-called a *strongly rpp semigroup* is an rpp semigroup in which every L_a^* contains a unique idempotent $a^\diamond \in L_a^* \bigcap E(S)$ such that $a^\diamond a = a$, where $E(S)$ is the set of idempotents of S. In fact, C-rpp semigroups are strongly rpp semigroups. There are many authors having been investigating various classes of strongly rpp semigroups (see, [3]-[6], [10]-[13], [16], [18], [19], [24] and [25]). A strongly rpp semigroup S is called a *left C-rpp semigroup* if $\mathcal{L}^*$ is a congruence and $eS \subseteq Se$ for all $e \in E(S)$. Guo-Shum-Zhu [19] established a construction of left C-rpp semigroups. After then, there are many works on this aspect (see, [6], [12], [14], [15] and [18]).

On the other hand, a strongly rpp semigroup S is called a *right C-rpp semigroup* if $\mathcal{L}^* \vee \mathcal{R}$ is a congruence and $Se \subseteq eS$ for all $e \in E(S)$ (see [16]). In the letters, right C-rpp semigroup is a dual of left C-rpp semigroup. But, in fact, there are less dual properties between left C-rpp semigroups with right C-rpp semigroups (for detail, see [12] and [25]). Recently, Guo-Ren-Shum [11] established a construction method of right C-rpp semigroups by making use of the developed wreath product. Recently, Guo-Jun-Zhao [10] defined *pseudo-C-rpp semigroups* by weakening the condition that $Se \subseteq Se$ for any idempotent e in the definition of right C-rpp semigroups. They obtained a construction of pseudo-C-rpp semigroups.

Recall from [23], an *orthocryptic group* is defined as a completely regular semigroup in which the relation $\mathcal{H}$ is a congruence and whose set of idempotents forms a band. Because strongly rpp semigroups are generalizations of completely regular semigroups in the range of rpp semigroups, it is naturally posed which kind of rpp semigroups the analogue of orthocryptic groups should be. This is the aim of this paper.

2 Preliminaries

In this section, we recall some known results used in the sequel. To begin with, we provide some results on $\mathcal{L}^*$ and dual for the relation $\mathcal{R}^*$.

Lemma 2.1 [2] *Let S be a semigroup and $e^2 = e, a \in S$. The the following statements are equivalent:*

(1) $a\mathcal{L}^*e$.

(2) $ae = a$ *and for all* $x, y \in S^1$, $ax = ay$ *implies that* $ex = ey$.

It is well known that $\mathcal{L}^*$ is a right congruence. In general, $\mathcal{L} \subseteq \mathcal{L}^*$. And, when a, b are regular elements of S, $a\mathcal{L}^*b$ if and only if $a\mathcal{L}b$. For convenience, we denote by $E(S)$ the set of idempotents of S and by $Reg(S)$ the set of regular elements of S. We use a^* [resp. $a^\dagger$] to denote the typical idempotents related to a by $\mathcal{L}^*$ [resp. $\mathcal{R}^*$]. And, K_a stands for the $\mathcal{K}$-class of S containing a if $\mathcal{K}$ is an equivalence on S.

In order to research rpp semigroups, Guo-Shum-Zhu [19] introduced so-called (ℓ)-Green's relations:

$$\begin{array}{rcl} a\mathcal{L}^{(\ell)}b & \Leftrightarrow & Ker a_\ell = Ker b_\ell, \quad i,e., \quad \mathcal{L}^{(\ell)} = \mathcal{L}^*; \\ a\mathcal{R}^{(\ell)}b & \Leftrightarrow & Im a_\ell = Im b_\ell, \quad i.e., \quad \mathcal{R}^{(\ell)} = \mathcal{R}; \\ \mathcal{D}^{(\ell)} & = & \mathcal{L}^{(\ell)} \vee \mathcal{R}^{(\ell)}; \\ \mathcal{H}^{(\ell)} & = & \mathcal{L}^{(\ell)} \bigcap \mathcal{R}^{(\ell)}; \\ a\mathcal{J}^{(\ell)}b & \Leftrightarrow & J^{(\ell)}(a) = J^{(\ell)}(b), \end{array}$$

where a_ℓ is the inner left translation of S^1 determined by a and $J^{(\ell)}(a)$ is the smallest ideal of S containing a and such that $J^{(\ell)}(a)$ is the union of some $\mathcal{L}^{(\ell)}$-classes.

Let I, Λ be arbirary non-empty sets and $P = (p_{\lambda i})$ a $\Lambda \times I$-matrix with entries $p_{\lambda i}$ in the units of the monoid M. If we write $S = M \times I \times \Lambda$ and define a multiplication on S by

$$(a, i, \lambda)(b, j, \mu) = (ap_{\lambda j}b, i, \mu),$$

then, same as the Rees theorem, we can easily verify that S is a semigroup. We now denote this semigroup S by $\mathcal{M}(M; I, \Lambda; P)$ and call it the Rees matrix semigroup over M with sandwich matrix P. Then $Reg(\mathcal{M}(M; I, \Lambda; P))$ (the set of all regular elements of $\mathcal{M}(M; I, \Lambda; P)$) is a completely simple subsemigroup of S. Also, it is trivial to see that $E(S) = \{(p_{\lambda i}^{-1}, i, \lambda) | i \in I, \lambda \in \Lambda\}$ is the set of idempotents of the semigroup $S = \mathcal{M}(M; I, \Lambda; P)$, where $p_{\lambda i}^{-1}$ is the corresponding inverse element of $p_{\lambda i}$ in M. It is easy to see that $Reg(\mathcal{M}(M; I, \Lambda; P)) = \mathcal{M}(Reg(M); I, \Lambda; P)$. The following lemma is an immediate consequence of [23, Theorem III.5.2, p.142].

Lemma 2.2 *Let $\mathcal{M} = \mathcal{M}(M; I, \Lambda; P)$ be the Rees matrix semigroup over M with sandwich matrix P. If $E(\mathcal{M})$ is a band, then every entry of P is the identity of M. Moreover, $\mathcal{M}$ is the direct product of M and the rectangular band E, where $E = I \times \Lambda$ with the multiplication: $(i, \lambda)(j, \mu) = (i, \mu)$.*

Lemma 2.3 (1) [8, Theorem 2.3] *If $D^{(\ell)}$ is a $\mathcal{D}^{(\ell)}$-class of a strongly rpp semigroup S, then $D^{(\ell)}$ is a $\mathcal{D}^{(\ell)}$-simple strongly rpp semigroup and the set of regular elements of $D^{(\ell)}$ is a completely simple semigroup.*

(2) [8, Theorem 3.4] *A semigroup is a $\mathcal{D}^{(\ell)}$-simple strongly rpp semigroup if and only if it is isomorphic to some Rees matrix semigroup $\mathcal{M}(M; I, \Lambda; P)$ over a left cancellative monoid M with the sandwich matrix P.*

For a strongly rpp semigroup S, we define a relation on S as

$$a\mathcal{R}^{(\dagger)}b \Longleftrightarrow a^{\diamond}\mathcal{R}b^{\diamond}.$$

Evidently, $\mathcal{R}^{(\dagger)}$ is an equivalence on S. Furthermore, we define $\mathcal{H}^{(\dagger)} := \mathcal{R}^{(\dagger)} \cap \mathcal{L}^{(\ell)}$.

Definition 2.4 *A strongly rpp semigroup is called an* ***orthodox cryptic rpp semigroup****, in short,* ***orthocryptic rpp semigroup****, if* (1) *$E(S)$ (the set of idempotents of S) forms a band; and* (2) *the relation $\mathcal{H}^{(\dagger)}$ is a congruence, that is, S is cryptic rpp.*

Theorem 2.5 *The following conditions are equivalent for a strongly rpp semigroup S:*

(1) *S is orthocryptic.*

(2) *For any $a, b \in S$, $a^{\diamond}b^{\diamond} = (ab)^{\diamond}$.*

Proof. (1) $\Rightarrow$ (2) Assume S is orthocryptic. Then for any $a, b \in S$, $a^{\diamond}b^{\diamond} \in E(S)$, thereby $(a^{\diamond}b^{\diamond})^{\diamond} = a^{\diamond}b^{\diamond}$, thus by Lemma 2.5, $(ab)^{\diamond} = (a^{\diamond}b^{\diamond})^{\diamond} = a^{\diamond}b^{\diamond}$.

(2) $\Rightarrow$ (1) Assume (2) holds. Note that $e = e^{\diamond}$ for any $e \in E(S)$. We always have $a^{\diamond}b^{\diamond} = (a^{\diamond}b^{\diamond})^{\diamond}$ if $E(S)$ is a band, in this case, $(ab)^{\diamond} = (a^{\diamond}b^{\diamond})^{\diamond}$ for any $a, b \in S$, hence $\mathcal{H}^{(\dagger)}$ is a congruence. So, it suffices to verify that $ef \in E(S)$ for any $e, f \in E(S)$. In fact, since $e = e^{\diamond}$ and $f = f^{\diamond}$, we have $ef = e^{\diamond}f^{\diamond} = (ef)^{\diamond} \in E(S)$, as required. □

Lemma 2.6 *If S is an orthocryptic rpp semigroup, then $\mathcal{D}^{(\ell)}$ is a semilattice congruence on S.*

Proof. By hypothesis, the set E of idempotents of S is a band and by Lemma 2.3 (1), for all $a, b \in S$,

$$a\mathcal{D}^{(\ell)}b \Leftrightarrow a^{\diamond}\mathcal{D}^{E}b^{\diamond}.$$

Now, by Theorem 2.5, $ab\mathcal{D}^{(\ell)}a^{\diamond}b^{\diamond}$ and thus $\mathcal{D}^{(\ell)}$ is a congruence on S. Note that $\mathcal{D}$ is a semilattice congruence on E, therefore we easily observe that $\mathcal{D}^{(\ell)}$ is a semilattice congruence on S. □

Recall from [13] that a semigroup is called a *left cancellative plank* if it is isomorphic to the direct product of a left cancellative monoid and a rectangular band. Based on Lemmas 2.2 and 2.6, we have

Proposition 2.7 *Every orthocryptic rpp semigroup is a semilattice of left cancellative planks.*

3 Characterizations

In this section we will establish some characterizations of orthodox cryptic rpp semigroups.

To begin with, we introduce some notations.

Let B be a band such that $B = (Y; E_\alpha)$, which is the semilattice decomposition of B into rectangular bands E_α with $\alpha \in Y$. If $e \in E_\alpha$, then we can write $E(e) = E_\alpha$. Also, we write $E_\alpha \leq E_\beta$ if $E_\alpha E_\beta \subseteq E_\alpha$.

Proposition 3.1 *If S is orthocryptic rpp semigroup, then the relation $\xi_S = \{(x, y) \in S \times S : (\exists e, f \in E(y^\diamond)\ x = eyf\}$ is indeed a C-rpp semigroup congruence on S preserving $\mathcal{L}^{(\ell)}$-classes.*

Proof. Assume S is orthocryptic. By proposition 2.7, we may assume S is the semilaatice Y of left cancellative planks $M_\alpha \times E_\alpha$ with $\alpha \in Y$, where M_α is a left cancellative monoid with identity 1_α and $E_\alpha = I_\alpha \times \Lambda_\alpha$ is a rectangular band. Clearly, $E(S) = \cup_{\alpha \in Y} 1_\alpha \times E_\alpha$. Write $B = \cup_{\alpha \in Y} E_\alpha$. Define an operation on B by: for $(i_\alpha, \lambda_\alpha) \in E_\alpha,\ (i_\beta, \lambda_\beta) \in E_\beta$,

$$(i_\alpha, \lambda_\alpha)(i_\beta, \lambda_\beta) = (i, k)$$

if $(1_\alpha, i_\alpha, \lambda_\alpha)(1_\beta, i_\beta, \lambda_\beta) = (1_{\alpha\beta}, i, k)$. It is easy to check that B is a band isomorphic to $E(S)$. On the other hand, we easily know that for any $(m, x) \in M_\alpha \times E_\alpha$, $(m, x)^\diamond = (1_\alpha, x)$ and $E((m, x)^\diamond) = \{1_\alpha\} \times E_\alpha$. This means that for any $(m, x) \in M_\alpha \times E_\alpha$ and $(n, y) \in M_\beta \times E_\beta$,

$$(*) \qquad (m, x)\ \xi_S\ (n, y) \Leftrightarrow \alpha = \beta,\ m = n.$$

Lemma 3.2 *If $(m, x), (m, y) \in M_\alpha \times E_\alpha$ and $(n, z) \in M_\beta \times E_\beta$, then*

(1) *The first components of $(m, x)(n, z)$ and $(m, y)(n, z)$ are the same.*

(2) *The first components of $(n, z)(m, x)$ and $(n, z)(m, y)$ are the same.*

Proof. We only prove (1) and (2) can be similarly proved. Let $(m, y)(n, z) = (q, v)$ and $(m, x)(n, z) = (p, u)$. We divide the proof into two steps:

Step A. Assume $\alpha \leq \beta$. Clearly, $(q, v), (p, u) \in M_\alpha \times \Lambda_\alpha$. Compute

$$\begin{aligned}
(q, x) &= (1_\alpha, x)(q, v)(1_\alpha, x) = (1_\alpha, x)(m, y)(n, z)(1_\alpha, x) = (1_\alpha, x)(m, y)(s, w)\\
&= (1_\alpha, x)(m, y)(1_\alpha, y)(1_\alpha, w)(s, w)\\
&= (1_\alpha, x)(m, y)[(1_\alpha, y)(1_\alpha, x)(1_\alpha, w)](s, w)\\
&= [(1_\alpha, x)(m, y)(1_\alpha, y)(1_\alpha, x)][(1_\alpha, w)(s, w)\\
&= (m, x)(s, w) = (m, x)(n, z)(1_\alpha, x) = (p, u)(1_\alpha, x)\\
&= (p, ux),
\end{aligned}$$

where $(n, z)(1_\alpha, x) = (s, w) \in M_\alpha \times E_\alpha$. It follows that $p = q$.

Step B. Obviously, $(q, v), (p, u) \in M_{\alpha\beta} \times E_{\alpha\beta}$. By Step A, we have

$$(n, z)(1_{\alpha\beta}, u) = (a, k), \qquad (n, z)(1_{\alpha\beta}, v) = (a, l)$$
$$(1_{\alpha\beta}, u)(m, x) = (b, f), \qquad (1_{\alpha\beta}, v)(1_\alpha, y)(m, x) = (b, g)$$

where $a, b \in M_{\alpha\beta}, k, l, f, g \in E_{\alpha\beta}$. So,

$$(p, \mu) = (m, x)(n, z) = [(1_{\alpha\beta}, u)(m, x)][(n, z)(1_{\alpha\beta}, u)] = (ba, fk).$$

On the other hand, since $(1_\alpha, y)(1_{\alpha\beta}, g) \in E_{\alpha\beta}$, we have $(1_{\alpha\beta}, g)(1_\alpha, y)(1_{\alpha\beta}, l) = (1_{\alpha\beta}, g)(1_{\alpha\beta}, l)$, so that

$$\begin{aligned}(m, y)(n, z) &= [(1_{\alpha\beta}, v)(m, y)][(n, z)(1_{\alpha\beta}, v)] = (1_{\alpha\beta}, v)(m, y)(a, l)\\ &= (1_{\alpha\beta}, v)(1_\alpha, y)(m, x)(1_\alpha, y)(a, l) = (b, g)(1_\alpha, y)\\ &= (b, g)(1_{\alpha\beta}, g)(1_\alpha, y)(1_{\alpha\beta}, l)(a, l) = (b, g)(1_{\alpha\beta}, g)(1_{\alpha\beta}, l)(a, l)\\ &= (ba, gl).\end{aligned}$$

We complete the proof. □

We need still prove

Lemma 3.3 *If $x\ \xi_S\ y$ then $E(x^\diamond) = E(y^\diamond)$ and $y^\diamond x y^\diamond = y$.*

Proof. If $x\xi_S y$, then $x = eyf$ for some $e, f \in E(y^\diamond)$, and therefore $xf = eyff = eyf = x$. By $x\mathcal{L}^\diamond x^\diamond$, this shows that $x^\diamond = x^\diamond f$, hence $E(x^\diamond) \leq E(f) = E(y^\diamond)$. On the other hand, by $x = eyf$, we have

$$y^\diamond x y^\diamond = y^\diamond eyfy^\diamond = (y^\diamond e y^\diamond)y(y^\diamond f y^\diamond) = y^\diamond y y^\diamond = y,$$

that is, $y\ \xi_S;\ x$. By applying similar arguments to $y\xi_S x$, $E(y^\diamond) \leq E(x^\diamond)$. Now, $E(y^\diamond) = E(x^\diamond)$. Here we have indeed verified that ξ_S is symmetric. □

Obviously, ξ_S is reflexive. To show that ξ_S is an equivalence on S, it remains to prove ξ_S is transitive. To see this, let $x, y, z \in S$ and $x\ \xi_S\ y$, $y\ \xi_S\ z$, then by Lemma 3.3, $E(x^\diamond) = E(y^\diamond) = E(z^\diamond)$ and $y^\diamond x y^\diamond = y$, $z^\diamond y z^\diamond = z$, thus $z^\diamond y^\diamond x y^\diamond z^\diamond = z$. Note that $z^\diamond y^\diamond, y^\diamond z^\diamond \in E(x^\diamond)$. Therefore $z\xi_S x$, and whence ξ_S is transitive, as required.

By Lemma 3.2, ξ_S is a congruence on S. Now, by $(*)$, S/ξ_S is a semilattice Y of the semigroups $M_\alpha \times E_\alpha/\xi_S$. Note that $M_\alpha \times E_\alpha/\xi_S$ is isomorphic to M_α. Thus S/ξ_S is a semilattice of left cancellative monoids, and whence S/ξ_S is a C-rpp semigroup. Now, it is easy to see that ξ_S preserves $\mathcal{L}^{(\ell)}$-classes. □

For any semigroup, we denote the smallest semilattice congruence by $\mathcal{N}$. Let $S = (Y; S_\alpha)$ and $T = (Z; T_\beta)$, where S_α and T_β are the $\mathcal{N}$-classes of S

and T, respectively. Assume there exists an isomorphism ξ of Y onto Z. The set $\cup_{\alpha\in Y} S_\alpha \times T_{\alpha\xi}$ is a subsemigroup of the direct product of S and T, called the *spined product of S and T relative to ξ* (for detail, see [22, I.8.12. Spined products, p.17]).

Theorem 3.4 *A semigroup S is an orthocryptic rpp semigroup if and only if S is a spined product of a band and a C-rpp semigroup.*

Proof. ($\Rightarrow$) Suppose S is an orthocryptic rpp semigroup. Then $E(S)$ is a band and by Proposition 3.1, $S/_{\xi_S}$ is a C-rpp semigroup. Let $E(S) = (Y; E_\alpha)$ be the semilattice decomposition of $E(S)$ into rectangular bands E_α. Note that the restriction of ξ_S to $E(S)$ is just $\mathcal{D}^{E(S)}$, therefore $E(S/\xi_S)$ is isomorphic to Y. We will identity $E(S/\xi_S)$ with Y. Now, by Lemma 3.3, we may assume $S/_{\xi_S} = (Y; M_\alpha)$ is the semilattice decomposition of S/ξ_S into left cancellative monoids M_α. Hence the spined product of $S/_{\xi_S}$ and the band $E(S)$ with respect to the semilattice Y is $M = \cup_{\alpha\in Y}(M_\alpha \times E_\alpha)$, where the multiplication on M is defined by $(m,i)\cdot(n,j) = (mn, ij)$, and mn and ij are the semigroup products of m and n in $S/_{\xi_S}$ and i and j in $E(S)$, respectively.

Define

$$\theta:\ S \to M;\ s \mapsto (s\xi_S, s^\diamond).$$

Since ξ_S preserves $\mathcal{L}^{(\ell)}$-classes, θ is well defined. It remains to prove that θ is a semigroup isomorphism. We divide the proof into the following steps:

Step A. If $(s\xi_S, s^\diamond) = (t\xi_S, t^\diamond)$, then $s\xi_S = t\xi_S$ and $s^\diamond = t^\diamond$. By the first equality, there exist $e, f \in E(t^\diamond)$ such that $s = etf$. But $E(t^\diamond)$ is a rectangular band, now we have

$$s = s^\diamond s s^\diamond = t^\diamond etf t^\diamond = t^\diamond e t^\diamond t t^\diamond f t^\diamond = t^\diamond t t^\diamond = t$$

and so θ is injective.

Step B. For any $(a,i) \in M$, then $a \in S/_{\xi_S}$ and so there exists $x \in S$ such that $x\xi_S = a$. By the definition of M, $x^\diamond \in E(i)$ and further $ix^\diamond i = i$. This shows $(ixi)\xi_S = x\xi_S = a$. On the other hand, by Theorem 2.5, $(ixi)^\diamond = i^\diamond x^\diamond i^\diamond = i$. Thus $(ixi)\theta = (a,i)$. Therefore θ is surjective.

Step C. By Theorem 2.5 and Proposition 3.1, we have

$$(st)\theta = ((st)\xi_S, (st)^\diamond) = (s\xi_S t\xi_S, s^\diamond t^\diamond) = (s\xi_S, s^\diamond)(t\xi_S, t^\diamond) = (s)\theta(t)\theta$$

Thus θ is a homomorphism. Therefore θ is an isomorphism, as needed.

($\Leftarrow$) Suppose that S is a spined product of a band E and a C-rpp semigroup M. Then we may assume $S = (Y; M_\alpha \times \Lambda_\alpha)$, where $M = (Y; M_\alpha)$ is the semilattice decomposition of the C-rpp semigroup M into left cancellative monoids M_α and $E = (Y; E_\alpha)$ the semilattice decomposition E into rectangular bands E_α. Obviously, $E(S) = \cup_{\alpha\in Y} 1_\alpha \times E_\alpha$, where 1_α is the identity of

M_α. By hypothesis, the mapping defined by $(1_\alpha, i) \to i$ is an isomorphism of $E(S)$ onto E, thus $E(S)$ is a band.

We next prove that S is rpp. Let $x = (a_\alpha, i_\alpha) \in M_\alpha \times I_\alpha$ and $y = (1_\alpha, i_\alpha) \in M_\alpha \times I_\alpha$. Obviously, $xy = x$. If $z = (b_\beta, j_\beta) \in M_\beta \times I_\beta$, $w = (c_\gamma, k_\gamma) \in M_\gamma \times I_\gamma$ such that $xz = xw$, then $\alpha\beta = \alpha\gamma = \delta$ and $(a_\alpha b_\beta, i_\alpha j_\beta) = (a_\alpha c_\gamma, i_\alpha k_\gamma)$. Comparing the components of the second equality, we have $a_\alpha b_\beta = a_\alpha c_\gamma$ and $i_\alpha j_\beta = i_\alpha k_\gamma$. Note that $a_\alpha b_\beta = (1_{\alpha\beta} a_\alpha)(1_\alpha b_\beta)$ and $a_\alpha c_\gamma = (1_{\alpha\beta} a_\alpha)(1_\alpha c_\gamma)$, and since $M_{\alpha\beta}$ is a left cancellative monoid, we obtain that $1_\alpha b_\beta = 1_\alpha c_\gamma$. Thus

$$yz = (1_\alpha b_\beta, i_\alpha j_\beta) = (1_\alpha c_\gamma, i_\alpha k_\gamma) = yw.$$

On the other hand, if $x = xw$, then $(x =)\ xy = xw$ and by the above proof, $(y =)\ yy = yw$. Now we have proved that for all $z, w \in S^1$, $xz = xw$ implies that $yz = yw$. Consequently, $x\mathcal{L}^*y$ and whence S is rpp.

Evidently, $yx = x$. Now let $z = (1_\beta, j_\beta) \in E(M_\beta \times I_\beta)$ such that $x\mathcal{L}^*z$ and $zx = x$. Then $y\mathcal{L}^*z$ and so $y\mathcal{L}z$. This shows that $yz = y$ and $zy = z$, thereby $\alpha = \beta$ and $yzy = z$. But $E(M_\alpha \times I_\alpha)$ is a rectangular band, $yzy = z$ can imply that $y = z$. Therefor S is a strongly rpp semigroup in which $(a_\alpha, i_\alpha)^\diamond = (1_\alpha, i_\alpha)$ for $(a_\alpha, i_\alpha) \in M_\alpha \times I_\alpha$.

Finally, if $x = (a_\alpha, i_\alpha) \in M_\alpha \times E_\alpha$ and $y = (b_\beta, j_\beta) \in M_\beta \times I_\beta$, then

$$xy = (a_\alpha b_\beta, i_\alpha j_\beta) \in M_{\alpha\beta} \times E_{\alpha\beta}$$

and by the forgoing arguments, $(xy)^\diamond = (1_{\alpha\beta}, i_\alpha j_\beta)$, $x^\diamond = (1_\alpha, i_\alpha)$ and $y^\diamond = (1_\beta, j_\beta)$. Thus

$$x^\diamond y^\diamond = (1_\alpha 1_\beta, i_\alpha j_\beta) = (1_{\alpha\beta}, i_\alpha j_\beta) = (xy)^\diamond.$$

and by Theorem 2.5, S is an orthocryptic rpp semigroup. □

References

[1] J.B. Fountain, Right pp momids with central idempotents, *Semigroup Forum* 13(1977), 9-23.

[2] J.B. Fountain, Abundant semigroups, *Proc. London Math. Soc.* 44(1982), 103-129.

[3] X.J. Guo, The structure of PI-strongly rpp semigroups, *Chinese Science Bulletin* 41(1996), 1647-1650.

[4] X.J. Guo, The structure of Nbe-rpp semigroups, *Northeastern Math. J.* 16(2000), 398-404.

[5] X.J. Guo and Y.Q. Guo, The translational hull on a strongly right type-A semigroup, *Scinece in China (ser. A)* 29(2000), 1002-1008.

[6] X.J. Guo, Y.Q. Guo and K.P. Shum, Semispined product structure of left C-a semigroups, In: *Semigroups* (ed: K.P. Shum, Y.Q. Guo et al.), Springer-Verlag, Singapore, 1998, pp. 157-166.

[7] X.J. Guo, Y.Q. Guo and K.P. Shum, Left abundant semigroups, *Communications in Algebra* 32(2004), 2061-2085.

[8] X.J. Guo, Y.Q. Guo and K.P. Shum, Rees matrix representation theorem for $\mathcal{D}^{(\ell)}$-simple strongly rpp semigroups, *Asian-European J. Math.* 1(2008), 215-223.

[9] X.J. Guo, Y.Q. Guo and K.P. Shum, Super rpp semigroups (submitted)

[10] X.J. Guo, Y.B. Jun and M. Zhao, Pseudo-C-rpp semigroups, To appear in *Acta Mathematica Sinica (English Ser.), 2009.*

[11] X.J. Guo, C.C. Ren and K.P. Shum, Dual Wreath Product structure of Right C-rpp Semigroups, *Algebra Colloquium* 14(2007), 285-294.

[12] X.J. Guo and K.P. Shum, On left cyber groups, *Inter. Math. J.* 5(2004), 705-717.

[13] X.J. Guo, K.P. Shum and Y.Q. Guo, Perfect rpp semigroups, *Communications in Algebra* 29(2001), 2447-2460.

[14] X.J. Guo and A.J. Wu, A note on left C-rpp semigroups, *J. Jiangxi Normal Univ. (Natural Sci.)* 29(2005), 283-286.

[15] X.J. Guo, M. Zhao and K.P. Shum, Wreath product structure of left C-rpp semigroups, *Algebra Colloquium* 15(2008), 101-108.

[16] Y.Q. Guo, On the dual of left C-rpp semigroups, *Chinese Science Bulletin* 42(1997), 1599-1603.

[17] Y.Q. Guo, Weakly left C-semigroups, *Chinese Science Bull.* 41(1996), 462-467.

[18] Y.Q. Guo, X.M. Ren and K.P. Shum, Another structure of left C-semigroups, *Advances in Mathematics (China)*, 24(1)(1995), 39-43.

[19] Y.Q. Guo, K.P. Shum and P.Y. Zhu, The structure of left C-rpp semigroups, *Semigroup Forum* 50(1995), 9-23.

[20] Y.Q. Guo, K.P. Shum and P.Y. Zhu, On quasi-C-semigroups and some special subclasses, *Algebra Colloquium* 6(1999), 105-120.

[21] J.M. Howie, *An introduction to semigroup theory*, Academic Press, London, 1976.

[22] M. Petrich, *Lecture in semigroups*, Akademic Verlag, Berlin, 1977.

[23] M. Petrich and N.R. Reilly, *Commpletely regular semigroups*, John Willey & sons, INC, New York, 1999.

[24] K.P. Shum, X.J. Guo and X.M. Ren, (ℓ)-Green's relations and perfect rpp semigroups, In: *Proceedings of the* 3^{th} *Asian Mathematical Conference 2000* (ed: T. Sunada, W.S. Polly and L. Yang), 604-613.

[25] K.P. Shum and X.M. Ren, The structure of right C-rpp semigroups, *Semigroup Forum* 68(2004), 280-292.

Received: April, 2009