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On Dimensions of Spectral Spaces of Finite Lattices

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Abstract

In this note we prove that the Krull dimension of a finite distributive lattice $(L, \leq)$ coincide with the inductive dimension and the derived dimension of the spectral space of L.

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Let $(P, \leq)$ be a poset. For $x \in P$, we define $\downarrow x = \{y \in P : y \leq x\}$ and $\uparrow x = \{y \in P : x \leq y\}$. For $A \subseteq P$, $\downarrow A = \cup\{\downarrow a : a \in A\}$ and $\uparrow A = \{\uparrow a : a \in A\}$. The set A is called a *lower set* if $A = \downarrow A$ and A is called an *upper set* if $A = \uparrow A$. Let us recall that a lattice $(L, \vee, \wedge, \leq)$ is a poset in which every pair of elements has a join and a meet; we denote a lattice simply by L or by $(L, \leq)$ if we want to emphasize on order structure of L. A lattice is *complete* if any of its subsets has join and meet.

The classical Stone Representation Theorem for Boolean algebra states that any Boolean algebra B is associated to a totally disconnected (zero-dimensional) compact Hausdorff topological space *spec* B, called *Stone space* and conversely, the lattice of *clopen*[1] subsets of any topological space X is a Boolean algebra, *clop* X. Moreover, a Boolean algebra B and its counterpart, *clop* (*spec* B), are isomorphic.

Recall that in a complete distributive lattice $(L, \leq)$, an element $x \in L$ is called *prime* if for every $a, b \in L$ $a \wedge b \leq x$ implies $a \leq x$ or $b \leq x$; $x \in L$ is

[1] A subset which is closed and open.

called *join-irreducible* if $x = a \vee b$ implies $x = a$ or $x = b$. Denote by *spec* L the set of all prime elements of L other than 1. We have $y = \inf(\uparrow y \cap spec\, L)$ where $y \in L$.

Let $h(a) = \{p \in spec\, L : a \leq p\} = \uparrow a \cap spec\, L$, then $h(a)$ is called the *hull* of a. Let *spec* L be given a topology with closed sets $\{h(a) : a \in L\}$. With respect to this topology there is a lattice isomorphism

$$\begin{array}{rcl} L & \longrightarrow & \Gamma(spec\, L) = \text{closed subsets of } spec\, L \\ x & \mapsto & \uparrow x \cap spec\, L \end{array}$$

So if $X\,(L) =$ *spec* L is equipped with the above topology, then L and $\Gamma\,(X\,(L))$ are isomorphic lattices.

For more details see ([2], *Chapter* V).

Let X be a topological space. For any ordinal α define *derived set* of order α by $X_0 = X$, $X_1 = X'$, the limit points of X, and $X_{\alpha\div1} = X'_\alpha$; if α is limit ordinal then $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$. Hence $X_0 \supseteq X_1 \supseteq \ldots \supseteq X_\alpha \supseteq X_{\alpha+1} \supseteq \ldots$. The smallest ordinal α such that $X_\alpha = \emptyset$ is called *derived dimension* of X and it is denoted by $d'\,(X)$ [3].

Now let L be a finite lattice. We shall consider three different ways of approaching the notion of dimension in L. Luckily, all those turn out to be equal.

First the the Krull dimension of L.

Definition 0.1 *For a finite distributive lattice L we define the krull dimension of L by*

$$K\dim(L) = Sup\,\{\, n : \exists \text{ a prime chain of length } n \text{ in } L\}\ .$$

Now, it is possible to topologize L by using $\overline{\{y\}} = \{x \in L : x \leq y\}$ as a sub-basis for the closed sets, i.e., by taking the lower sets to be the closed sets. On the other hand, let X be any finite T_0-space .Then for $x \in X$, we can form the minimal open set containing x by U_x to be the intersection of all open sets containing x. Clearly, these minimal neighborhoods are basis for the topology of X. For distinct x, y in X, we can define a partial order relation $\leq$ on X by setting $x \leq y$ when $U_y \subseteq U_x$. This order is the same as the natural order, called the *specialization order*, defined by letting $x \leq y$ if and only if $x \in \overline{\{y\}}$. Indeed, if x and y are two distinct elements of X then $y \in U_x$ implies that for any open set O containing x we have $(O - \{x\}) \cap \{y\} \neq \emptyset$. Therefore $x \in \overline{\{y\}}$.

Conversely, $x \in \overline{\{y\}}$ implies that for every open set O of x we have $O \cap \{y\} \neq \emptyset$. So, $y \in O$. Now, by choosing $O = U_x$, we get $U_x \supseteq U_y$ or $x \leq y$. Since closed sets in a finite topological space are just unions of closures of points, we see that any T_0-space X can be completely determined by the (unique) specialization order defined on X. Moreover, a map between finite T_0-spaces is continuous if and only if it is order preserving. Hence the *category* of finite T_0 topological spaces and continuous maps is isomorphic to the category of finite posets and order-preserving maps.

As we observed at above, any finite distributive lattice L is order isomorphic to the lattice of the closed subsets of $X(L) = spec\, L$. Notice that points of $X(L) = spec\, L$ are prime elements of L. Let us denote by $X^*(L)$ the dual space of the space $X(L) = spec\, L$, i.e., the topological space having closed sets as open sets of $X^*(L)$. Hence $X^*(L)$ is determined by the reverse order of the specialization order of L.

Now, let X be a topological space and $x \in X$. Then, with respect to the natural order, $\{x\}$ is *minimal* if $\{x\}$ is closed in X and it is *maximal* if $\{x\}$ is open in X. In the latter case, x is an isolated point of the space X.

Definition 0.2 *The inductive dimension of a topological space X, $ind\, X$, is defined inductively by taking $ind\, X$ to be -1 if $X = \emptyset$. For $X \neq \emptyset$ define $ind\, X \leq n$ if for each x and for each open set O_x containing x, there exists an open set $U \subseteq O_x$ where $x \in U$ such that the boundary of U, $\partial U = \overline{U} \cap \overline{(X - U)}$, has inductive dimension at most $n-1$; $ind\, X = n$ if $ind\, X \leq n$ and $ind\, X$ is not less than or equal to $n-1$.*

Let X be a finite T_0-space. Then $\mathcal{B}$, the collection of all minimal neighborhoods for each of its points, called a *minimal basis* for X. Note that, any other basis for X contains $\mathcal{B}$. Therefore if $indX \leq n$, then $ind\,(\partial B_x) \leq n-1$, $x \in X$.

For the simple proof of the next proposition and many other properties of finite topological spaces we refer the reader to [1].

Proposition 0.3 *Let X be a finite T_0-space with minimal basis $\mathcal{B}$. Then $ind\, X \leq n$ if and only if $ind\ (\partial B_x) \leq n-1$, for every $B_x \in \mathcal{B}$.*

Our main theorem is a consequece of the following proposition and its corollary.

Proposition 0.4 *For a finite space X with $ind\, X = n$, the length of the longest chain of closures of points $\overline{\{x_0\}} \subseteq \overline{\{x_1\}} \subseteq \ldots \subseteq \overline{\{x_k\}}$ is n.*

Proof:
The proof is by induction on n. It is clearly true for $n = -1$. Assume that the proposition holds for dimension $n-1$. Now let $ind\, X = n-1$. We are going to prove that X has a chain of length n. Let x be a point with smallest open set U containg x such that $ind\ (\partial U) = n-1$. By hypothesis there exists a chain of length n where $\overline{\{x_1\}} \subseteq \overline{\{x_2\}} \subseteq \ldots \subseteq \overline{\{x_n\}}$ in $\partial U = \overline{U} \cap \overline{(X-U)}$.

On the other hand, $x_1 \in \overline{U} = \{y \in X : \exists x_0 \in U \ni y \leq x_0\}$, hence there exists $x_0 \in U$ with $x_1 \leq x_0$. But x_1 is not in U, therefore $x_1 \neq x_0$ and since for $x, y \in X$, $\overline{\{x\}} \subset \overline{\{y\}}$ is equivalent to $y < x$, where $y < x$ means $x \leq y$ and $x \neq y$. Then $x_0 > x_1 > x_2 > ... > x_n$ and, consequently, $\overline{\{x_0\}} \subset \overline{\{x_1\}} \subset \ldots \subset \overline{\{x_n\}}$, which is a chain of length n in X.

Now it remains to prove that there exists no chain of closures of points in X of length $> n$. Suppose that $\overline{\{x_0\}} \subset \overline{\{x_1\}} \subset \ldots \subset \overline{\{x_k\}}$ is a chain of length k in X. Let U_k be the smallest open set containing x_k. Then *ind* $(\partial U_k) \leq n-1$. Since U_k is the smallest open set of x_k and $x_{k-1} < x_k$, we have that $\overline{\{x_{k-1}\}} \cap U_{x_k} = \emptyset$. Therefore $\overline{\{x_0\}} \subset \overline{\{x_1\}} \subset \ldots \subset \overline{\{x_{k-1}\}}$ is a chain in $\partial U_{x_k} = \overline{U_{x_k}} \cap \overline{(X - U_{x_k})} = \overline{U_{x_k}} - U_{x_k}$. By induction hypothesis we have $k-1 \leq n-1$. That is, $k \leq n$. This completes the proof of the proposition. □

Now we have have the following corollary .

Corollary 0.5 *For a finite space X, ind $X =$ ind X^* .*

Proof:
The routine verification is left to the reader. Note that $\overline{\{x\}} \subset \overline{\{y\}}$ is equivalent to $y < x$ for $x, y \in X$. □

Using this we can prove the following result for finite lattices.

Theorem 0.6 *Let L be a finite distributive lattice endowed with its natural topology. Then*

$$ind\ (spec\ L) = d'\,(spec\ L) = K\dim(L)\,.$$

Proof:
Recall that we have denoted the dual of $X\,(L)$ by $X^*\,(L)$ and $x \in X\,(L) = spec L$ is maximal (with respect to the natural order) if $\{p \in spec\ L : p \geq x\} = \{x\}$ is open in $X\,(L) = spec\ L$. But then $\{x\}$ is a closed set in $X^*\,(L)$. So, if M_0 is the set of all maximal points of $X\,(L)$ and M_j is the set of all maximal points of $X\,(L) \setminus \bigcup_{i=0}^{j-1} M_i$, then $X\,(L) = \bigcup_{i=0}^{n} M_i = \bigcup_{i=0}^{n} M_j^o$, where $ind\,X\,(L) = n$ and M_j^o is the set of all closed points of $X^*\,(L) \setminus \bigcup_{i=0}^{j-1} M_i^o$. By the last proposition, $ind\,X\,(L) = ind\,X^*\,(L) = n$. Now recall that for each i, M_i is a set of isolated (maximal) points and hence a discrete subspace of X. Then, by Proposition **??**, $ind\,M_i = 0$ and $M_{n+1} = \emptyset$. Consequently, we have the relation $X_0 \supseteq X_1 \supseteq \ldots \supseteq X_n \supseteq X_{n+1}$ which implies that $d'\,(spec\,L) = n$.

On the other hand, since $X\,(L) = \bigcup_{i=0}^{n} M_i = \bigcup_{i=0}^{n} M_j^o$ and $ind\,M_i = 0$ for each i, the maximal length of chains of prime elements of $X\,(L)$ has length n.

Indeed, each discrete subset M_i can have at most one point in such a chain of prime elements. This implies that each prime chain of $X(L)$ has length at most n. Thus $ind\, X = d'(X) = K\dim(L)$, where $X = X(L) = spec\, L$. Therefore for finite distributive lattices all of those dimensions coincide. $\square$

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