International Mathematical Forum, 4, 2009, no. 41, 2041 - 2050

# On Operators of Weighted Composition on Spaces of Orlicz-Functions

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**Abstract.** In this paper we characterize the weighted composition Operators on Orlicz function spaces and we also make an efforts to characterize compactness, invertibility and Fredholmness of these operators.

**Mathematics Subject Classification:** Primary 47B33; 47B38 Secondary, 46E30

**Keywords:** Weighted composition operator, compact operator, invertible operator, Orlicz functions space

## 1. Introduction

Let $(\Omega, S, \mu)$ be a $\sigma$-finite measure space and let $L^{\phi}(\mu) = \{f | f : \Omega \to C$ is measurable such that $\int_{\Omega} \phi(\alpha f) d\mu < \infty$, for some $\alpha > 0\}$. Then $L^{\phi}(\mu)$ is a Banach space under the norm,

$$||f||_{\phi} = \inf\{\epsilon > 0 : \int_{\Omega} \phi(\frac{|f|}{\epsilon}) d\mu \leq 1\}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous convex function which satisfy the following

(i) $\phi(x) = 0$ if and only if x = 0
(ii) $\lim_{x\to\infty} \phi(x) = \infty$
Such a function $\phi$ is known as a Young's function.
Let $u : \Omega \to C$ be a measurable function and $v : \Omega \to \Omega$ be a non-singular measurable transformation. Then a bounded linear transformation, $m_{u,v} : L^{\phi}(\mu) \to L^{\phi}(\mu)$ defined by

$$(m_{u,v}f)(x) = u(x).f(v(x))$$

is called a weighted composition operator induced by the pair $(u, v)$. If we take $u(x) = 1$, the constant one function on $\Omega$, we write $m_{u,v}$ as $T_v$ and call it a composition operator induced by $v$. In case $v(x) = x$ for some $x \in \Omega$, we write $m_{u,v}$ as $m_u$ and call it a multiplication operator induced by $u$.
By $B(L^{\phi}(\mu))$, we denote the set of all bounded linear operators from $L^{\phi}(\mu)$ into itself.
If $v$ is a non-singular measurable transformation, then the measure $\mu v^{-1}$ is absolutely continuous with respect to the measure $\mu$. Hence by Radon Nikodym derivative theorem there exists a positive measurable function $w$ such that $\mu(v^{-1}(E)) = \int w d\mu$ for some $E \in S$. The function $w$ is called the Radon Nikodym derivative of the measure $\mu v^{-1}$ with respect to the measure $\mu$. It is denoted by $w = \frac{d\mu v^{-1}}{d\mu}$.

Let $(\Omega, S, \mu)$ be a $\sigma$-finite measure space and $S_0 \subset S$ be a $\sigma$-finite subalgebra. Then the conditional expectation $E(.|S_0)$ is defined as a linear transformation from certain S-measurable function spaces ($i.e. L^1, L^2$ etc) into their$S_0$-measurable counterparts. In particular the conditional expectation with respect to the $\sigma$-algebra $v^{-1}(S)$ is a bounded projection from $L^p(\Omega, S, \mu)$ onto $L^p(\Omega, v^{-1}(S), \mu)$. We denote this transformation by E. The transformation E has the following properties.

1. $E(f.gov) = E(f).(gov)$
2. If $f \geq g$ almost everywhere, then $E(f) \geq E(g)$ almost everywhere
3. $E(1) = 1$
4. E(f) has the form $E(f) = gov$ for exactly one $\sigma$- measurable function g. In particular $g = E(f)ov^{-1}$ is a well defined measurable function.
5. $|E(fg)|^2 \leq (E|f|^2)(E|g|^2)$
6. For $f > 0$ almost everywhere, $E(f) > 0$ almost everywhere.
7. If $\phi$ is a convex function, then $\phi(E(f)) \leq E(\phi(f))$ $\mu$- almost every where.

For more details on Orlicz spaces one can refer to ([7], [8], [9], [11]), where as the classes of weighted composition operators on some function spaces are considered by ([1], [2], [3], [4], [5], [6], [10]). In this paper we plan to study the weighted composition operators on Orlicz function spaces and we take $(\Omega, S, \mu)$ be a $\sigma$-finite non-atomic measure space.

## 2. On Operators of Weighted composition on spaces of Orlicz functions

**Theorem 2.1.** *Let $u : \Omega \to C$ and $v : \Omega \to \Omega$ be two mappings.Then $m_{u,v} : L^{\phi}(\mu) \to L^{\phi}(\mu)$ is a bounded operator if and only if there exists a constant $M > 0$ such that*

$$w(x)\phi(|(E(|u|ov^{-1})(x)||y|) \leq \phi(M|y|) \qquad (i)$$

*for $\mu$-almost all $x \in \Omega$ and $y \in C$.*

*Proof.* Suppose the condition (i) is true. Then for every $f \in L^{\phi}(\mu)$. We have

$$\begin{aligned}
\int_{\Omega} \phi(\frac{|M_{u,v}f|}{M||f||_{\phi}})d\mu &= \int_{\Omega} \phi(\frac{|u.fov|}{M||f||_{\phi}})d\mu \\
&= \int_{\Omega} \phi(\frac{|E(|u|)ov^{-1}f|}{M||f||_{\phi}})d\mu v^{-1} \\
&\leq \int_{\Omega} w(x)\phi(\frac{|E(|u|)ov^{-1}(x)|f(x)|}{M||f||_{\phi}})d\mu \\
&\leq \int_{\Omega} \phi(\frac{|f|}{||f||_{\phi}})d\mu \\
&\leq 1
\end{aligned}$$

so for every $f \in L^{\phi}(\mu), ||M_{u,v}f||_{\phi} \leq M||f||_{\phi}$. Hence $m_{u,v}$ is a bounded linear operator.
Conversely, If the condition (i) is not satisfied, then for every positive integer n, there exists a measurable set $G_n$ of $\Omega$ and some $y_n \in C$ such that
$G_n = \{x \in \Omega : w(x)\phi(|(E(|u|)ov^{-1})(x)||y_n|) > \phi(2^n n^2 y_n)\}$
is a measurable set of positive measure. Since $\mu$ is non atomic, so we can choose a disjoint sequence of measurable sets $H_n$ such that $H_n \subset G_n$ and

$$\mu(H_n) = \frac{\phi(|y_1|)}{2^n \phi(n^2|y_n|)}$$

Let $f = \sum_{n=1}^{\infty} a_n \chi_{H_n}$, where $a_n = n y_n$

Consider

$$\begin{aligned}
\int_{\Omega} \phi(\alpha f) d\mu &= \sum_{n=1}^{\infty} \int_{\Omega} \phi(\alpha a_n) \chi_{H_n} d\mu \\
&= \sum_{n=1}^{n_0} \phi(\alpha a_n) \mu(H_n) + \sum_{n \geq n_o} \phi(\alpha a_n) \mu(H_n) \\
&\leq \sum_{n=1}^{n_0} \phi(\alpha a_n) \mu(H_n) + \sum_{n \geq n_o} \phi(\alpha a_n) | \frac{\phi(|y_1|)}{2^n \phi(n^2 |y_n|)} \\
&= \sum_{n=1}^{n_0} \phi(\alpha a_n) \mu(H_n) + \sum_{n \geq n_o} \phi(n^2 |y_n|) \frac{\phi(|y_1|)}{2^n \phi(n^2 |y_n|)} \\
&= \sum_{n=1}^{n_0} \phi(\alpha a_n) \mu(H_n) + \sum_{n \geq n_o} \frac{\phi(|y_1|)}{2^n} \\
&< \infty,
\end{aligned}$$

where $n_o$ is such that $n_o > \alpha$.

But

$$\begin{aligned}
\int \phi(\alpha m_{u,v} f) d\mu &= \int_{\Omega} w \phi(\alpha((E(|u|) o v^{-1} |f|)) d\mu \\
&\geq \sum_{n > m_o} \int_{H_n} w \phi(\frac{1}{n} |E(|u|) o v^{-1} a_n|) d\mu, \text{ where } \alpha > \frac{1}{m_0} \\
&\geq \sum_{n \geq m_0} \int_{H_n} w \phi(((|E(|u|) 0 v^{-1}) || y_n|) d\mu \\
&\geq \sum_{n \geq m_o} \phi(2^n . n^2 |y_n|) \mu(H_n) \\
&\geq \sum_{n \geq m_o} \phi(2^n . n^2 |y_n|) \frac{\phi(|y_1|)}{2^n \phi(n^2 |y_n|)} \\
&\geq \sum_{n \geq m_o} \phi(|y_1|) \\
&= \infty,
\end{aligned}$$

which contradicts the fact that $m_{u,v} f \in L^{\phi}(\mu)$. Hence the condition of the theorem must be true. □

**Theorem 2.2.** *Let $m_{u,v} \in B(L^{\phi}(\mu))$. Then $m_{u,v}$ is a compact operator if and only if $E(|u|) o v^{-1} . w = 0$ a.e.*

*Proof.* We first assume that $m_{u,v}$ is compact. For if $|E(|u|)ov^{-1}.w \neq 0$ a.e. then there exists $k > 0$ such that the set $G = \{x \in \Omega : |E(|u|)ov^{-1}(x)| > \frac{1}{k}\} \cap \{x : w(x) > \frac{1}{k}\}$ has a positive measure. From the non atomicness of the measure $\mu$, we can find measurable sets $G_n$ such that $G_{n+1} \subset G_n \subset G$, and $\mu(G_n) = \frac{b}{2^n}$, where $0 < b < \mu(G_n)$.
Let $e_n = \phi^{-1}(\frac{1}{\mu(G_n)})\chi_{G_n}$. Then $||e_n|| = 1$.
So that the sequence $\{e_n\}$ is bounded in $L^{\phi}(\mu)$. On the other hand for any $m, n \in N$, let m = 2n. Then $G_m \subset G_n$ and

$$\begin{aligned}\int_{\Omega} \phi(\frac{|\frac{1}{k^2}\chi_{(G_n-G_m)}|}{||\chi_{G_n}||_{\phi}||m_{u,v}e_n - m_{u,v}e_m||_{\phi}})d\mu &\leq \int_{\Omega} w\phi\left(\frac{|E(|u|)ov^{-1}|\chi_{e_n} - |E(|u|)ov^{-1}|\chi_{e_m}}{||m_{u,v}e_n - m_{u,v}e_m||_{\phi}}\right) d\mu \\ &= \int_{\Omega} \phi\left(\frac{|m_{u,v}e_n - m_{u,v}e_m|}{||m_{u,v}e_n - m_{u,v}e_m||_{\phi}}\right) d\mu \\ &\leq 1.\end{aligned}$$

Therefore

$$\begin{aligned}||m_{u,v}e_n - m_{u,v}e_m||_{\phi} &\geq \frac{1}{k^2}\frac{||\chi_{G_n-G_m}||_{\phi}}{||\chi_{G_n}||_{\phi}} \\ &> \frac{\epsilon}{k^2}\end{aligned}$$

for some $\epsilon > 0$. which shows that $\{m_{u,v}e_n\}$ does not contain a convergent subsequence. Therefore $m_{u,v}$ is not compact. This contradicts our hypothesis. Hence $E(|u|)ov^1.w = 0$ a.e.
Conversely, if $E(|u|)ov^{-1}.w = 0$ a.e., then $m_{u,v} = 0$ and so it is compact. □

**Corollary 2.3.** *No non- zero weighted composition operator on $L^{\phi}(\mu)$ is compact, when the underlying measure space is non- atomic.*

**Theorem 2.4.** *Let $m_{u,v} : L^{\phi}(\mu) \to L^{\phi}(\mu)$ be a bounded operator. Then $m_{u,v}$ has closed range if and only if there exists $\delta > 0$ such that*

$$w(x)\phi(|(E(|u|)ov^{-1})(x)||y|) \leq \phi(\delta|y|)$$

*for $\mu$-almost all $x \in s$, where $s =$ support $(E(|u|)ov^{-1}) \cap V(\Omega)$ and for each $y \in C$.*

*Proof.* Assume that the condition is true. Then for every $f \in L^{\phi}(\Omega \setminus s)$ and $\delta > 0$, we have

$$\begin{aligned}1 &\geq \int_{\Omega} \phi(\frac{|m_{u,v}f|}{||m_{u,v}f||_{\phi}}d\mu \\ &= \int_{\Omega} \phi(\frac{||u.fov|}{||m_{u,v}f||_{\phi}})d\mu \\ &= \int_{\Omega} w\phi(\frac{(E(|u|)ov^{-1})|f|}{||m_{u,v}f||_{\phi}})d\mu \\ &\geq \int_{\Omega} \phi(\frac{\delta|f|}{||m_{u,v}f||_{\phi}})d\mu\end{aligned}$$

This shows that $||m_{u,v}f||_\phi \geq \delta||f||_\phi$ for all $f \in L^\phi(\Omega \setminus s)$. So that $m_{u,v}$ has closed range, since Ker $m_{u,v} = L^\phi(\Omega \setminus s')$.
Conversely, suppose $m_{u,v}$ has closed range. Then there exists $\delta > 0$ such that $||m_{u,v}f||_\phi \geq \delta||f||_\phi$ for all $f \in L^\phi(\Omega \setminus s')$ $(i)$
Let $y \in C$ and let

$$\begin{aligned} E_n &= \{x \in s : \phi(\frac{1}{(n+1)^2}|y|) \\ &\leq w(x)\phi(|E(|u|)ov^{-1})(x)||y|) \\ &\leq \phi(\frac{1}{n^2}|y|) \end{aligned}$$

and $E = \{n : \mu(E_n) > 0\}$. Let $f = \sum_{n\in E} \phi^{-1}(\frac{1}{\mu(E_n)})m_{u,v}\chi_{E_n}$.

Then

$$\begin{aligned} \infty &> \sum_{n\in E} \frac{1}{n^2} \\ &\geq \sum_{n\in E} \int_{E_n} \phi(\frac{1}{n^2}\phi^{-1}(\frac{1}{\mu(E_n)}))d\mu \\ &\geq \sum_{n\in E} \int_{E_n} w\phi(\phi^{-1}(\frac{1}{\mu(E_n)})E(|u|)ov^{-1})d\mu \\ &= \sum_{n\in E} \int_{\Omega} \phi(\phi^{-1}(\frac{1}{\mu(E_n)})m_{u,v}\chi_{E_n}))d\mu \\ &= \int_\Omega \phi(f)d\mu \end{aligned}$$

Now for $h = \sum_{n\in E} \phi^{-1}(\frac{1}{\mu(E_n)})\chi_{E_n}$, we have

$$f = m_{u,v}\left(\sum_{n\in E} \phi^{-1}(\frac{1}{\mu(E_n)})\chi_{E_n}\right) = m_{u,v}h.$$

In view of inequality (i) we have, $||m_{u,v}h||_\phi \geq \delta||h||_\phi$. But

$$\begin{aligned}\infty &= \sum_{n\in E} 1 \\ &= \sum_{n\in E} \frac{1}{\mu(E_n)}\mu(E_n) \\ &= \sum_{n\in E}\int_{E_n} \phi(\phi^{-1}(\frac{1}{\mu(E_n)}))d\mu \\ &= \sum_{n=1}^{\infty}\int_{\Omega} \phi(\phi^{-1}(\frac{1}{\mu(E_n)})\chi_{E_n})d\mu \\ &= \int_{\Omega}\phi(h)d\mu,\end{aligned}$$

which is a contradiction. Hence E must be a finite set. In other words, there exists $n_0$ such that $\mu(E_n) = 0$ for all $n \geq n_0$. That is
$w(x)\phi(|(E(|u|)ov^{-1})(x)||y|) \geq \phi(\frac{1}{n_0^2}|y|) = \phi(\delta|y|)$ (say). □

**Theorem 2.5.** *Let $m_{u,v} \in B(L^\phi(\mu)$. Then $m_{u,v}$ is bounded away zero if and only if there exists $\delta > 0$ such that $w(x)\phi(|E(|u|)ov^{-1})(x)|y|) > \phi(\delta||y|)$ for $\mu$-almost all $x \in \Omega$ and for each $y \in C$.*

*Proof.* The proof is exactly similar to that of theorem 2.4. □

**Theorem 2.6.** *Let $m_{u,v} \in B(L^\phi(\mu))$. Then $m_{u,v}$ is an injective if and only if $u \neq 0$ a.e., and $v$ is surjection.*

*Proof.* If the conditions of the theorem are true, then we show that $m_{u,v}$ is injective. Suppose $m_{u,v}f = 0$. We show that $m_{u,v}$ is injective. Suppose $m_{u,v}f = 0$. Then u.fov = 0 which implies that $u(x)f(v(x)) = 0$ for $\mu$-almost all x. Therefore f(v(x)) = 0. But $v$ is surjective, hence f = 0. This shows that $m_{u,v}$ is injective.

Conversely, if the set $E = \{x : u(x) = 0\}$ is of positive measure, then for $0 \neq f \in L^\phi(\mu)$, we have $\chi_E f \in$ Ker $m_{u,v}$, so that $m_{u,v}$ has non-trivial kernel, which is a contradiction. Hence $u(x) \neq 0$ a.e.

Next if $v$ is not surjective, then taking any measurable set F of positive measure such that $F \subset \Omega \setminus v(\Omega)$, we have $\chi_F \in$ Ker $m_{u,v}$, which is again a contradiction. Hence $v$ must be surjective. □

**Theorem 2.7.** *Let $m_{u,v} \in B(L^\phi(\mu))$. Then $m_{u,v}$ has dense range if and only if $u \neq 0$ a.e., and $v^{-1}(S) = S$.*

*Proof.* We first assume that $m_{u,v}$ has dense range. Let $G = \{x : u(x) = 0\}$. If $\mu(G) > 0$, then choose a measurable subset H of G such that $0 < \mu(H) < 1$.

Consider
$< \chi_H, m_{u,v}f >= \int_H u(x)f(v(x))d\mu = 0$
which shows that $\chi_H \in (ranm_{u,v})^{\perp}$. This contradicts that $m_{u,v}$ has dense range. Hence $\mu(G) = 0$. Thus $u \neq 0$ a.e.

The converse of the theorem is obvious. □

**Theorem 2.8.** *Let $m_{u,v} \in B(L^{\phi}(\mu))$. Then $m_{u,v}$ is invertible if and only if*
*(i) $u \neq 0$, a.e.*
*(ii) $v$ is invertible*
*(iii) $w(x)\phi(|(E(|u|)ov^{-1})(x)||y|) \geq \Phi(\delta|y|)$ for $\mu$-almost all $x \in \Omega$ and for each $y \in C$.*

*Proof.* Suppose $m_{u,v}$ is invertible.Then clearly $u \neq 0$ a.e. and $v^{-1}(S) = S$. This implies that $v$ is injective. Also in view of theorem (2.5),
$w(x)\phi(|(E(|u|)ov^{-1})(x)||y|) \geq \Phi(\delta|y|)$ (i)
for $\mu$-almost all $x \in \Omega$ and for each $y \in C$ and for some $\delta > 0$. It follows from (i) that $w \neq 0$, a.e. So that $v$ is surjective. Thus $v$ is invertible.

Conversely, assume that the conditions of the theorem are true. Then by theorem (2.5) and theorem (2.7), $m_{u,v}$ is bounded away from zero and has dense range. This shows that $m_{u,v}$ is invertible. □

**Theorem 2.9.** *Let $m_{u,v} \in B(L^{\phi}(\mu))$. Then $m_{u,v}$ is Fredholm if and only if $m_{u,v}$ is invertible.*

*Proof.* If $m_{u,v}$ is invertible, then obviously $m_{u,v}$ is Fredholm.

Conversely, suppose that $m_{u,v}$ is Fredholm. Then Ker $m_{u,v}$ and $L^{\phi}(\mu)/ranm_{u,v}$ are finite dimensional and ran $m_{u,v}$ is closed. If Ker$m_{u,v}$ is finite dimensional, then ker$m_{u,v} = 0$, otherwise it will be infinite dimensional. Therefore in view of theorem (2.6) $u \neq 0$. Suppose co-dim ran$m_{u,v} < \infty$. We claim that $m_{u,v}$ is onto. For if ran$m_{u,v}$ is not dense, then it is a proper closed subspace of $L^{\phi}(\mu)$ and for $f \in L^{\phi}(\mu) \setminus ranm_{u,v}$ there exists a continuous linear functional $h^* \in L^{\phi^*}(\mu)$ such that

$$h^*(m_{u,v}f) = \int (m_{u,v}f)h^* d\mu = 0$$

and

$$h^*(f) = \int fh^* d\mu = 1.$$

From the later equality Re$fh^* = 1$. Hence the set

$$G_\delta = \{x \in \Omega : Re(fh^*)(x) \geq \delta\}$$

has positive measure for some $\delta > 0$. Since $\mu$ is non atomic, we can find a sequence $\{G_n\}$ of measurable subsets of $G_\delta$ with $0 < \mu(G_n) < \infty$ such that $G_n \cap G_m = \phi$ for $m \neq n$. Let $h_n^* = \chi_{G_n} h^*$. Clearly $h_n^* \in L^{\phi^*}(\mu)$ and is non zero. Now for each $f \in L^{\phi}(\mu)$,

$$(m_{u,v}^* h_n^*)(f) = h_n^*(m_{u,v}f)$$

i.e.

$$\int \chi_{G_n} h^* m_{u,v} f d\mu = \int (m_{u,v} f \chi_{G_n}) g^* d\mu = 0$$

Thus $m^*_{u,v} h^*_n = 0$ for each n = 1, 2, 3, . . . . This implies that ker $m^*_{u,v}$ and hence $L^\phi(\mu) \setminus \overline{ranm_{u,v}}$ is infinite dimensional, which is a contradiction. Hence $m_{u,v}$ has dense range. So $m_{u,v}$ is bounded away from zero and has dense range. Therefore $m_{u,v}$ is invertible. □

**Theorem 2.10.** *Let $m_{u,v} \in B(L^\phi(\mu))$. Then $m_{u,v}$ is an isometry if and only if $(E(|u|)ov^{-1}) = 1$ a.e. $[\mu v^{-1}]$.*

*Proof.* Suppose $m_{u,v}$ is an isometry. For $f \in (L^\phi(\mu))$,
Consider

$$\begin{aligned} ||f||_{\phi,\mu} &= ||m_{u,v} f||_{\phi,\mu} \\ &= inf\{\epsilon > 0 : \int_\Omega \phi(\frac{|ufov|}{\epsilon}) d\mu \leq 1\} \\ &= inf\{\epsilon > 0 : \int_\Omega \phi\left(\frac{E(|u|ov^{-1})|f|}{\epsilon}\right) d\mu v^{-1} \leq 1\} \\ &= ||m_{u,v} f||_{\phi,\mu v^{-1}} \ (i) \end{aligned}$$

It follows from (i) that $||f|| = ||m_{u,v}||_{\phi,\mu v^{-1}}$ holds if and only if $E(|u|)ov^{-1} = 1$ a.e. $[\mu v^{-1}]$.

□

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**Received: January, 2009**