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Degree of Approximation of Lipschitz Function

By Product Summability Method

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Abstract

In this paper author have been determined the degree of approximation of certain function belonging to Lip α class by (C,1)(E, q) means of its Fourier series.

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1:Definition and notations$-$ Let $f(t)$ be periodic with period 2π and integrable in the Lebesgue sense .The Fourier series $f(t)$ is given by

$$f(t)\ \frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos nt+b_n\sin nt\right). \qquad (1)$$

A function $f \in Lip\alpha$ if

$$f(x+t)-f(x)=O\left(\left|t^{\alpha}\right|\right) \quad \text{for} \quad 0<\alpha\leq 1. \qquad (2)$$

The degree of approximation of a function $f: R \to R$ by a trigonometric polynomial t_n of order n is defined by Zygmund [1] (1968; P.114)

$$\|t_n-f\|_{\infty}=\sup\{\lfloor t_n(x)-f(x)\rfloor : x\in R\}. \qquad (3)$$

If $\left(E,q\right)=E_n^q=\dfrac{1}{\left(1+q\right)^n}\displaystyle\sum_{K=0}^{n} q^{n-k}s_k \to s \ as\ n\to\infty$ **.**

Then an infinite series $\sum_{k=o}^{\infty}u_k$ with the partial sums s_n is said to be summable (E, q) to the definite number s. (Hardy [4])

The series $\sum_{k=0}^{\infty}u_k$ is said to be (C,1) summable to s if

$$(C,1)=\frac{1}{(n+1)}\sum_{k=0}^{n}s_k \to s\ as\ n\to\infty.$$

The (C,1) transform of the (E, q) transform defines the (C,1)(E, q) transform of the partial sums s_n of the series $\sum_{k=0}^{\infty}u_k$.

Thus if $\left(CE\right)_n^q=\dfrac{1}{\left(n+`1\right)}\displaystyle\sum_{k=0}^{n}E_k^q \to s\ as\ n\to\infty$ (4)

where E_n^q denotes the (E, q) transform of s_n , then the series $\sum_{k=0}^{\infty}u_k$ is said to be summable (C,1)(E, q) means or simply summable (C,1)(E, q) to s.

We shall use following notation:

$$\phi(t)=f\left(x+t\right)+f\left(x-t\right)-2f\left(x\right).$$

2.Main theorem

The degree of approximation of functions belonging to Lipα by ceasaro means and by Nörlund means has been discussed by a number of researchers like Alexits[5], Sahney and Goel [3],Chandra [10] ,Qureshi [6], Qureshi and Nema [7], Lal [9], Rhoades[2]. Recently Lal and Kushwaha [11] have determined the degree of approximation of conjugate of functions by lower triangular matrix means. But till now no work seems to have been done to obtain the degree of approximation of the function belonging to Lipα by (C,1)(E, q) product means of its fourier series .In an attempt to make study in this direction,one theorem on the degree of approximation of function of Lipα class by product summability mean of the form (C,1)(E, q) has been determined as following.

Theorem. If $f : R \to R$ is 2π periodic, Lebesgue intigrable on $[-\pi, \pi]$ and belonging to the Lipschitz class then the degree of approximation of f by the (C,1)(E, q) product means of its Fourier series satisfies for n = 0,1,2 …

$$\left\| (CE)_n^q(x) - f(x) \right\|_\infty = O\left((n+1)^{-\alpha}\right) \qquad \text{for} \quad 0<\alpha<1 .$$

3. Lemmas: For the proof of our theorem, following lemmas are required:

Lemma: 1 Let $M_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left[\frac{1}{(q+1)^k} \sum_{r=0}^{k} \binom{k}{r} q^{k-r} \frac{\sin(r+1/2)t}{\sin(t/2)} \right]$

then $M_n(t) = O(n+1)$ for $0<t<(n+1)^{-1}$.

Proof: using $\sin nt \le n \sin t$ for $0<t<(n+1)^{-1}$

then
$$M_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left[\frac{1}{(q+1)^k} \sum_{r=0}^{k} \binom{k}{r} q^{k-r} \frac{(2r+1)\ \sin(t/2)}{\sin(t/2)} \right]$$

$$\le \frac{1}{(n+1)} \sum_{k=0}^{n} \left[\frac{1}{(q+1)^k} (2k+1) \sum_{r=0}^{k} \binom{k}{r} q^{k-r} \right]$$

$$= \frac{1}{(n+1)} \sum_{k=0}^{n} (2k+1) \qquad \left(\because \sum_{r=0}^{k} \binom{k}{r} q^{k-r} = (1+q)^k\right)$$

$$= O(n+1).$$

Lemma : 2 $\quad M_n(t) = O(1/t),\ \textit{for}\ (n+1)^{-1} < t < \pi$.

Proof-
$$M_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left[\frac{1}{(q+1)^k} \sum_{r=0}^{k} \binom{k}{r} q^{k-r} \frac{\sin(r+1/2)t}{\sin(t/2)} \right]$$

Now applying Jordan's lemma $\sin(t/2) \geq (t/\pi)$ and sin kt ≤ 1 *for* $(n+1)^{-1} \leq t \leq \pi$

we get
$$M_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left[\frac{1}{(q+1)^k} \sum_{r=0}^{k} \binom{k}{r} q^{k-r} \frac{1}{(t/\pi)} \right]$$

$$= O\left(\frac{1}{t}\right). \qquad \left(\because \sum_{r=0}^{k} \binom{k}{r} q^{k-r} = (1+q)^k\right)$$

Proof of theorem: The n^{th} partial sum $S_n(x)$ of the series (1) at t=x is written as

$$S_n(x) = f(x) + \frac{1}{2\pi} \int_0^{\pi} \phi(t) \frac{\sin(n+1/2)t}{\sin(t/2)} dt$$

so that (E, q) means of the series (1) are

$$E_n^q(x) = \frac{1}{(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} S_k(x)$$
$$= f(x) + \frac{1}{2\pi(q+1)^n} \int_0^{\pi} \frac{\phi(t)}{\sin(t/2)} \left(\sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin(k+1/2)t \right) dt.$$

Therefore (C,1)(E, q) means of the series (1) are

$$(CE)_n^q(x) = \frac{1}{(n+1)} \sum_{k=0}^{n} E_k^q(x) \qquad (n = 0,1,2,3...)$$

$$= f(x) + \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left\{ \frac{1}{(q+1)^k} \int_0^{\pi} \frac{\phi(t)}{\sin(t/2)} \left(\sum_{r=0}^{k} \binom{k}{r} q^{k-r} \sin(r+1/2)t \quad dt \right) \right\}$$

$$= f(x) + \int_0^{\pi} \phi(t) M_n(t) dt \qquad (5)$$

where
$$M_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left[\frac{1}{(q+1)^k} \sum_{r=0}^{k} \binom{k}{r} q^{k-r} \frac{\sin(r+1/2)t}{\sin(t/2)} \right].$$

So
$$(CE)_n^q(x) - f(x) = \int_0^{\pi} \phi(t) M_n(t) dt$$

$$= \left(\int_0^{1/(n+1)} + \int_{1/(n+1)}^{\pi} \right) \phi(t) \ M_n(t) \ dt$$

$$= \ I_1 + I_2 . \qquad (6)$$

Now $\qquad I_1 = \int_0^{1/(n+1)} \phi(t) \ M_n(t) \ dt$

by lemma (1) $\quad |I_1| \le \int_0^{1/(n+1)} \phi(t) \ O(n+1) \ dt$

$$= O(n+1) \int_0^{1/(n+1)} |t^{\alpha}| \ dt$$

$$= O(n+1) \left[\frac{t^{\alpha+1}}{\alpha+1} \right]_0^{1/(n+1)}$$

$$= \ O\left((n+1)^{-\alpha} \right). \qquad (7)$$

And $\qquad I_2 = \int_{1/(n+1)}^{\pi} \phi(t) \ M_n(t) \ dt$

by lemma (2)

$$|I_2| \le \int_{1/(n+1)}^{\pi} \phi(t) \ O\left(\frac{1}{t}\right) \ dt$$

$$= \int_{1/(n+1)}^{\pi} |t^{\alpha}| \ O\left(\frac{1}{t}\right) \ dt$$

$$= \int_{1/(n+1)}^{\pi} t^{\alpha-1} \ dt$$

$$= \left[\frac{t^{\alpha}}{\alpha} \right]_{1/(n+1)}^{\pi}$$

$$= \frac{1}{\alpha} \left[\pi^{\alpha} - \frac{1}{(n+1)^{\alpha}} \right]$$

$$\le \ \frac{1}{\alpha} \left[\pi^{\alpha} + \frac{1}{(n+1)^{\alpha}} \right]$$

$$= O\left((n+1)^{-\alpha}\right). \tag{8}$$

Now combining from (6) to (8),

$$\left|(CE)_n^q(x) - f(x)\right| = O\left((n+1)^{-\alpha}\right) \quad for \quad 0 < \alpha < 1.$$

Thus

$$\left\|(CE)_n^q(x) - f(x)\right\|_\infty = \sup_{-\pi \le x \le \pi} \left|(CE)_n^q(x) - f(x)\right|$$

$$= O\left((n+1)^{-\alpha}\right) \; for \; 0 < \alpha < 1.$$

This completes the proof of the theorem.

3. The following corollary can be derived from our theorem.

Corollary: If $f : R \to R$ is 2π periodic Lebesgue integrable on $[-\pi, \pi]$ and belonging to the Lipschitz class then the degree of approximation of f by (C,1)(E,1) product means of its Fourier series is given by

$$\left\|(CE)_n^1 - f(x)\right\|_\infty = O\left((n+1)^{-\alpha}\right) \; for \; 0 < \alpha < 1.$$

Note : An independent proof of the corollary can be derived by taking q=1 along the same line as the theorem.

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