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Fibonacci Lengths and Schur Multipliers of Non-Abelian Groups of Order less than 32

R. Golamie

Department of Mathematics, Islamic Azad University Tabriz Branch, Tabriz, Iran
rahmat_golamie@yahoo.co.uk

H. Doostie

Mathematics Department, Tarbiat Moallem University, 49 Mofateh Ave., Tehran 15614, Iran
doostih@saba.tmu.ac.ir

K. Ahmadidelir

Department of Mathematics, Islamic Azad University Tabriz Branch, Tabriz, Iran
kdelir@gmail.com

Abstract

Every finite group may be considered as a homomorphic image of some Fibonanci group $F(r, n)$. A definite method has been found to calculate n in this paper by using the notion of Fibonacci length. For all of the non-abelian groups of order less 32 the Fibonacci lengths together with the Schur multipliers have been calculated in this paper by providing certain sequences of integers. This study will give us a very simple proofs for the infiniteness of certain Fibonanci groups as well.

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1 Introduction

For every positive integers r and n where, $r \leq n$, the Fibonacci group $F(r, n)$ is defined to be

$$F(r,n) =< a_1, a_2, ..., a_n \mid a_1a_2...a_r = a_{r+1}, a_2...a_ra_{r+1} = a_{r+2}, \ldots >,$$

where indices are reduced modulo n. These groups and their generalizations are studied by many authors during the years and for a detailed consideration one may see [1,2,3]. The Fibonacci length of a non-abelian finite group $G =< a_1, ..., a_r >$ with respect to the generating set $A = \{a_1, \ldots, a_r\}$, is also defined to be the least integer t such that the sequence

$$x_i = a_i, (i = 1, ..., r), \quad x_i = \prod_{j=1}^{r} x_{i-r-1+j}, (i \geq r+1)$$

is periodic of the period t. Usually we use the notation $LEN_A(G)$ for t. For more information on the Fibonacci length of groups one may consult the references [4,5,6]. The following proposition is an essential relationship between the Fibonacci length and the Fibonacci groups.

Proposition 1.1. For a non-abelian finite group $G =< a_1, \ldots, a_r >$ where, $A = \{a_1, ..., a_r\}$ and $n = LEN_A(G)$, the group G is an epimorphic image of $F(r, n)$.

Proof. Considering the Fibonacci sequence of elements of G as

$$x_1 = a_1, x_2 = a_2, ..., x_r = a_r, x_{r+1} = x_1 x_2 ... x_r, x_{r+2} = x_2 x_3 ... x_r x_{r+1}, ... \quad (1)$$

we get $n > r$. Since $n = LEN_A(G)$ then $x_{n+1} = x_1, x_{n+2} = x_2, ..., x_{n+r} = x_r$. Consequently, the relations of $F(r, n)$ that is, the relations

$$\begin{aligned} x_1 x_2 ... x_r &= x_{r+1}, \\ x_2 x_3 ... x_{r+1} &= x_{r+2} \\ x_{n-1} x_n x_1 ... x_{r-2} &= x_{r-1} (= x_{n+r-1}), \\ x_n x_1 x_2 ... x_{r-1} &= x_r (= x_{n+r}), \end{aligned} \quad (2)$$

hold in the group F(r,n). This proves that G is a homomorphic image of F(r,n).

2 Calculations

All of the non-abelian groups of order less than 32 will be considered in this section and appropriate Fibonacci group $F(r, n)$ will be find for each group as noted in Section 1. Following [1] we call a non-abelian group G a $Z-$metacyclic group if G' and $\frac{G}{G'}$ are abelian. Also a $Z-$metacyclic group is called a $ZS-$metacyclic group if all of its Sylow subgroups are abelian. Our method of computation is based on using GAP[7] by performing certain computer procedures.

To calculate the Schur multiplier of a finitely presented group $G =< X \mid R >$, which will be denoted by $M(G)$ we will use the following proposition. The schur multiplier is defined to be $M(G) = \frac{F' \cap \bar{R}}{[F,\bar{R}]}$ where, $F = F(X)$ is the free group on X and $\bar{R}$ is the normal closure of R.

Proposition 2.1. For every $ZS-$metacyclic group G, $M(G) = 1$, and for every positive integer $n \geq 6$, $M(D_n) = 1$ or $M(D_n) = Z_2$ if n is odd either is even.

Proof. One may see [2].

The following proposition will be used to calculate the Fibonacci lengths of the dihedral groups $D_n =< a, b | a^2 = b^n = (ab)^2 = 1 >$ and the quaternion group Q_8.

Proposition 2.2. For every positive integer $n \geq 3$, $LEN_{\{a,b\}}(D_n) = 6$ and $LEN_{\{a,b\}}(Q_8) = 3$ where, $Q_8 =< a, b \mid a^4 = 1, a^2 = b^2, a^{-1}ba = b^{-1} >$.

Proof. One may see [5].

The studied groups consist of certain direct products. To calculate their Fibonacci lengths we have to consider the groups $G = Z_m \times D_n$ and in this context we first define the sequence $\{\gamma_k\}$ of numbers as follows:

$$\gamma_1 = 1, \gamma_2 = \gamma_3 = 0,\ \gamma_k = \gamma_{k-3} + \gamma_{k-2}\gamma_{k-1}, \quad (k \geq 4).$$

Then the following results are essential:

Lemma 2.3. Every element of the Fibonacci sequence

$$x_1 = c,\ x_2 = a,\ x_3 = b,\ x_k = x_{k-3}x_{k-2}x_{k-1},\ k \geq 4$$

of the elements of the group $G = Z_m \times D_n =< c > \times < a, b >$ may be presented by:

$$x_k = \begin{cases} c^{\gamma_k}, & \text{if } k \equiv 1 \text{ or } -3 (mod\ 8), \\ ac^{\gamma_k}, & \text{if } k \equiv \pm 2 (mod\ 8), \\ bc^{\gamma_k}, & \text{if } k \equiv 3 (mod\ 8), \\ abc^{\gamma_k}, & \text{if } k \equiv 4 (mod\ 8), \\ b^{-1}c^{\gamma_k}, & \text{if } k \equiv -1 (mod\ 8), \\ ab^{-1}c^{\gamma_k}, & \text{if } k \equiv 0 (mod\ 8), \end{cases}$$

Proof. The proof is almost easy by considering eight cases and using an induction method on k.

Proposition 2.4. For every positive integers $m, n \geq 3$ let $G = Z_m \times D_n =< c > \times < a, b >$ and $A = \{c, a, b\}$. Then, $l = LEN_A(G)$ if and only if l is divisible by 8 and satisfies all of the following equations:

$$\gamma_{l+1} \equiv 1(mod\ m),\ \gamma_{l+2} \equiv 0(mod\ m),\ \gamma_{l+3} \equiv 0(mod\ m).$$

Proof. By the definition of $LEN_A(G)$ we get $x_{l+1} = x_1 = c$, $x_{l+2} = x_2 = a$ and $x_{l+3} = x_3 = b$, and then by using the previous lemma one may get the result at once.

In the definite list of non-abelian groups of order less that 32, there are certain examples of the following families of groups, where, the calculations of the Fibonacci lengths have been performed individually by hand. These are the groups:

$$\begin{array}{l} < l, m, n >=< a, b, c \mid a^l = b^m = c^n = abc >, \\ < -l, m, n >=< a, b, c \mid a^{-l} = b^m = c^n = abc >, \\ < l, m \mid n >=< a, b \mid a^l = b^m, (ab)^n = 1 >, \\ < l, m \mid n; q > \simeq < m, l \mid n; q >=< a, b, c \mid a^l = b^m = c, (ab)^n = c^q = 1 > . \end{array}$$

For instance, for the groups $G =< 2, 2, n >=< a, b \mid a^n = b^2 = (ab)^2 >$ a hand calculation yields:

$$LEN_A(G) = \begin{cases} 3, & \text{if } n = 2, \\ 6, & \text{otherwise.} \end{cases}$$

We have collected our results in the following table and the concluded results on the infinite Fibonacci groups will be appeared at the end.

$\|G\|$	Symbol	Defining relation	$M(G)$	$LEN(G)$
6	$D_3 \cong S_3$	$a^3 = b^2 = (ab)^2 = 1$	1	6
8	D_4	$a^4 = b^2 = (ab)^2 = 1$	Z_2	6
8	$Q_8 \cong < 2, 2, 2 >$	$a^2 = b^2 = (ab)^2$	1	3
10	D_5	$a^5 = b^2 = (ab)^2 = 1$	1	6
12	D_6	$a^6 = b^2 = (ab)^2 = 1$	Z_2	6
12	A_4	$a^3 = b^2 = (ab)^3 = 1$	Z_2	16
12	$< 2, 2, 3 >$	$a^3 = b^2 = (ab)^2$	1	6
14	D_7	$a^7 = b^2 = (ab)^2 = 1$	1	6
16	$Z_2 \times D_4$	$a^2 = b^2 = c^3 = (ab)^2$	$Z_2 \times Z_2$	8
		$= (ac)^2 = (bc)^4 = 1$	$\times Z_2$	
16	$Z_2 \times Q_8$	$a^2 = b^2 = (ab)^2$	$Z_2 \times Z_2$	8
		$c^2 = 1$		
16	D_8	$a^8 = b^2 = (ab)^2 = 1$	Z_2	6
16	$< -2, 4 \| 2 >$	$a^2 = 1, aba = a^3$	1	6
16	$< 2, 2 \| 2 >$	$a^2 = 1, aba = a^{-3}$	1	6
16	$< 2, 2 \| 4; 2 >$	$a^4 = b^4 = 1, b^{-1}ab = a^{-1}$	Z_2	6
16	$< 4, 4 \| 2; 2 >$	$a^4 = b^4 = (ab)^2 =$	$Z_2 \times Z_2$	6
		$(a^{-1}b)^2 = 1$		
16	$< 2, 2, 2 >_2$	$a^2 = b^2 = c^2 = 1$	$Z_2 \times Z_2$	8
		$(abc) = (bca) = (cab)$		
16	$< 2, 2 \| 4 >$	$a^4 = b^2 = (ab)^2$	1	6
18	$Z_3 \times D_3$	$a^3 = b^6 = 1, b^{-1}ab = a^{-1}$	1	23
18	D_9	$a^9 = b^2 = (ab)^2 = 1$	1	6
18	$((3, 3, 3; 2))$	$a^2 = b^2 = c^2 = (abc)^2 =$	Z_3	4
		$(ab)^3 = (ac)^3 = 1$		
20	D_{10}	$a^{10} = b^2 = (ab)^2 = 1$	Z_2	6
20	$F^{2,1,-1}$	$a^2baba^{-1}b = b^2 = 1$	1	30
20	$< 2, 2, 5 >$	$a^5 = b2 = (ab)^2$	1	6
21		$a^3 = 1, a^{-1}ba = b^2$	1	36
22	D_{11}	$a^{11} = b^2 = (ab)^2 = 1$	1	6
24	$Z_2 \times A_4$	$a^3 = b^2 = (a^{-1}bab)^2 = 1$	Z_2	48
24	$Z_2 \times D_6$	$a^2 = b^2 = c^2 = (bc)^6 =$	$Z_2 \times Z_2$	24
		$(ab)^2 = (ca)^2 = 1$	$\times Z_2$	
24	$Z_3 \times D_4$	$a^{12} = b^2 = 1, bab = a^{-5}$	Z_2	12
24	$Z_3 \times Q_8$	$a^{12} = 1, b^2 = a^6,$	1	6
		$b^{-1}ab = a^7$		
24	$Z_4 \times D_3$	$a^{12} = b^2 = 1, bab = a^5$	Z_2	6
24	$Z_2 \times < 2, 2, 3 >$	$a^6 = b^4 = 1, b^{-1}ab = a^{-1}$	Z_2	24
24	D_{12}	$a^{12} = b^2 = (ab)^2 = 1$	Z_2	6

$\lvert G\rvert$	Symbol	Defining relation	$M(G)$	$LEN(G)$
24	S_4	$a^4 = b^2 = (ab)^3 = 1$	Z_2	6
24	$< 2, 3, 3 >$	$a^3 = b^2, (a^{-1}b)^3 = 1$	1	38
24	$< 4, 6\lvert 2, 2 >$	$a^4 = b^6 = (ab)^2$ $= (a^{-1}b)^2$	Z_2	6
24	$< -2, 2, 3 >$	$a^2 = b^2 = (ab)^3$	1	12
24	$< 2, 2, 6 >$	$a^6 = b^2 = (ab)^2$	1	6
26	D_{13}	$a^{13} = b^2 = (ab)^2 = 1$	1	6
27	$(3, 3\lvert 3, 3)$	$a^3 = b^3 = (ab)^3$ $= (a^{-1}b)^3 = 1$	$Z_3 \times Z_3$	8
27		$a^3 = 1, a^{-1}ba = b^{-2}$	1	24
28	$Z_2 \times D_7$	$a^{14} = b^2 = (ab)^2 = 1$	Z_2	6
28	$< 2, 2, 7 >$	$a^7 = b^2 = (ab)^2$	1	6
30	$Z_3 \times D_5$	$a^2 = 1, aba = b^4$	1	6
30	$Z_5 \times D_3$	$a^2 = 1, aba = b^{-4}$	1	60
30	D_{15}	$a^{15} = b^2 = (ab)^2 = 1$	1	6

As a result of the Propositions 1.1 and 2.2 we may give a very simple proof for the infiniteness of the Fibonacci group $F(2, 6)$. Indeed, the groups D_n are homomorphic images of $F(2, 6)$, for every $n \geq 3$.

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