International Mathematical Forum, 4, 2009, no. 43, 2143 - 2148

Two-Step Nilpotent Lie Algebras Attached to Graphs

Hamid-Reza Fanaï

Department of Mathematical Sciences
Sharif University of Technology
P.O. Box 11155-9415, Tehran, Iran
fanai@sharif.edu

Abstract

Consider a two-step nilpotent Lie algebra $\mathfrak{n}$ associated with a graph as introduced in [3] endowed with an inner product for which the vertices and the edges of the graph form an orthogonal basis. We show that there exists a rank-one Einstein metric solvable extension of $\mathfrak{n}$ if and only if the graph is positive. This generalizes the result of [6].

Mathematics Subject Classification: 53C30, 53C25.

Keywords: Einstein solvmanifold, nilpotent Lie algebra

1 Introduction

In this note, we consider examples of homogeneous Einstein manifolds of negative scalar curvature attached to graphs. The classical examples of Einstein metrics of negative scalar curvature are the symmetric spaces of non-compact type. Some other examples are known, e.g. [1], [7], [9], [11], [13]. It is interesting that all known examples of homogeneous Einstein manifolds of negative scalar curvature are isometric to Einstein Riemannian solvmanifolds, e.g. a simply connected solvable Lie group S together with a left invariant Einstein Riemannian metric g. In these examples, such a solvable Lie group S is a semi-direct product of an Abelian Lie group A with a nilpotent normal subgroup N, the nilradical. A left invariant Riemannian metric on a Lie group G will be identified with the inner product $\langle , \rangle$ determined on the Lie algebra $\mathfrak{g}$ of G, and the pair $(\mathfrak{g}, \langle , \rangle)$ will be referred to as a metric Lie algebra. If G is Einstein, we say that $\mathfrak{g}$ is an Einstein metric Lie algebra. A nilpotent Lie algebra $(\mathfrak{n}, [,])$ is said to be an Einstein nilradical if it admits an inner product $\langle , \rangle$ such that there exists an Einstein metric solvable extension of $(\mathfrak{n}, [,], \langle , \rangle)$, that is an Einstein metric solvable Lie algebra ($\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, [,], \langle,\rangle$) where $\mathfrak{a}$ is Abelian and $\mathfrak{a} \perp \mathfrak{n}$ such that the restrictions of the Lie bracket and the inner product of $\mathfrak{s}$ to $\mathfrak{n}$ coincide with the Lie bracket and the inner product of $\mathfrak{n}$, respectively. We use the same notation $[,]$ and $\langle,\rangle$ for $\mathfrak{s}$. We note that in the rank-one case, that is, $\dim \mathfrak{a} = 1$, such an Einstein extension if it exists, is unique. So rank-one Einstein solvmanifolds are completely determined by its metric nilpotent part. Recall that the study of Einstein solvmanifolds reduces to the rank-one case by [9]. These spaces have been studied well from different points of view. For example, this class of solvmanifolds includes the harmonic manifolds of negative Ricci curvature constructed by Damek and Ricci [2]. Also the class of spaces with volume preserving geodesic symmetries is related to such solvmanifolds [4] (see also [5] and [8]).

In [3], two-step nilpotent Lie algebras attached to graphs are considered, where their group of Lie automorphisms have been determined. In [6], we considered a two-step nilpotent Lie algebra $(\mathfrak{n}, [,])$ attached to a graph endowed with an inner product for which the vertices and the edges of the graph form an orthonormal basis and studied whether there exists a rank-one Einstein metric solvable extension of it. We described necessary and sufficient conditions of this problem in terms of the graph.

Recently in [12], among other results, the authors considered a more general problem in the same context. In fact, they considered a two-step nilpotent Lie algebra $(\mathfrak{n}, [,])$ attached to a graph and studied whether $\mathfrak{n}$ is an Einstein nilradical. As an application of results from geometric invariant theory, the authors proved the following result : a two-step nilpotent Lie algebra associated with a graph is an Einstein nilradical if and only if the graph is positive. The positivity of a graph means that a certain uniquely defined weighting on the set of edges is positive ([12]). The exact definition will be given later. In this note, with the same method as in [6], we show that a two-step nilpotent Lie algebra associated with a graph endowed with an inner product for which the vertices and the edges of the graph form an orthogonal basis is an Einstein nilradical if and only if the graph is positive. This is of course a more restricted version of the result of [12] but our method is more elementary. We recall the construction with graphs introduced in [3] and some notations in the next section.

2 Preliminary Notes

Let (V, E) be finite graph, where V is the set of vertices and E is the set of edges; equivalently E is a collection of unordered pairs of distinct vertices; the unordered pairs will be written in the form $\alpha\beta$, where $\alpha, \beta \in V$ and in this case we say that α and β are joined. Let $\mathfrak{v}$ be a vector space with V as a basis. Let $\mathfrak{z}$ be the subspace of $\wedge^2\mathfrak{v}$ the second exterior power of $\mathfrak{v}$, spanned by $\{\alpha \wedge \beta : \alpha, \beta \in V, \alpha\beta \in E\}$. Let $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$. Stipulating the conditions that for any $\alpha, \beta \in V$, $[\alpha, \beta] = \alpha \wedge \beta$ if $\alpha\beta \in E$ and 0 otherwise, and $[Z, X] = 0$ for all $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$ determines a unique Lie algebra structure on $\mathfrak{n}$ (see [3]). Clearly $(\mathfrak{n}, [,])$ is a two-step nilpotent Lie algebra. Thus the center of $(\mathfrak{n}, [,])$ is $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$ if and only if the graph has no any isolated vertex. We may assume that the graph has no isolated vertices since such vertices only determine an Abelian factor of $(\mathfrak{n}, [,])$. Two bases of $\mathfrak{v}$ and $\mathfrak{z}$ are $V = \{X_1, \dots, X_k\}$ and $\{\alpha \wedge \beta : \alpha, \beta \in V, \alpha\beta \in E\}$ respectively. Write the latter basis of $\mathfrak{z}$ as $\{Z_1, \dots, Z_r\}$ where each element of the basis is an edge $X_j X_{j'}$ with $1 \leq j < j' \leq k$.

Suppose that $\langle,\rangle$ is an arbitrary inner product on $\mathfrak{n}$ for which the following basis $b := \{X_1, \dots, X_k, Z_1, \dots, Z_r\}$ is orthogonal. We can then define a linear map $J : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ by

$$\langle J(Z)X, Y\rangle = \langle [X, Y], Z\rangle$$

for $X, Y \in \mathfrak{v}$ and $Z \in \mathfrak{z}$.

Consider now a metric solvable extension $(\mathfrak{s} = \mathbb{R}A \oplus \mathfrak{n}, [,], \langle,\rangle)$ of $\mathfrak{n}$ where the norm of A is one. We let $f := \mathrm{ad}_A : \mathfrak{n} \to \mathfrak{n}$. We can suppose that $\mathrm{tr}\, f > 0$. According to [9], if the associated Riemannian solvmanifold is Einstein, then without loss of generality, we can assume that f is symmetric relative to the inner product $\langle,\rangle$ and the matrix representation of f in an orthonormal basis of $\mathfrak{n}$ is in the following form ([9] and [10]) :

$$f = \begin{pmatrix} B_r & 0 \\ 0 & D_k \end{pmatrix}.$$

In fact the solvmanifold is Einstein if and only if there exists a negative constant μ such that we have the following relations :

$$\begin{array}{ll} (E_1) & -(\mathrm{tr}\, f)B + \frac{1}{4} J^* J = \mu\, \mathrm{Id}_r \\ (E_2) & -(\mathrm{tr}\, f)D + \frac{1}{2}\sum_{i=1}^{r} J^2(z_i) = \mu\, \mathrm{Id}_k \\ (E_3) & J(B(\cdot)) = J(\cdot)D + DJ(\cdot) \end{array}$$

where $\{z_1, \dots, z_r\}$ is an orthonormal basis of $\mathfrak{z}$ and the endomorphism $J^* J$ of $\mathfrak{z}$ is represented by the $r \times r$ matrix $(-\mathrm{tr}({}^t J(z_i) J(z_j)))_{i,j}$. In the next section we consider these relations.

3 Main Result

Suppose that $(\mathfrak{n}, [,], \langle,\rangle)$ is a metric two-step nilpotent Lie algebra associated with a graph as described above. Suppose that there exists an Einstein metric solvable extension $(\mathfrak{s} = \mathbb{R}A \oplus \mathfrak{n}, [,], \langle,\rangle)$. We let $f := \mathrm{ad}_A : \mathfrak{n} \to \mathfrak{n}$ and suppose that $\mathrm{tr}\, f > 0$. Without loss of generality, we can assume that f is symmetric relative to the inner product $\langle , \rangle$. Let $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ as above. We may assume that the graph has no isolated vertices so the center of $(\mathfrak{n}, [,])$ is $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$. As an orthogonal basis of $\mathfrak{n}$ we have $b = \{X_1, \dots , X_k, Z_1, \dots , Z_r\}$.

It is clear that we can assume that the basis b is an orthonormal basis of $\mathfrak{n}$ if we modifies the Lie bracket $[,]$ on $\mathfrak{n}$ as following with some suitable constant $a_1, \dots , a_r \in \mathbb{R} \setminus \{0\}$: $[X_j, X_{j'}] = a_i Z_i$ if $Z_i = X_j X_{j'}$ is an edge of the graph with $1 \leq j < j' \leq k$, $1 \leq i \leq r$ and zero otherwise. We use the same notation $[,]$ for this modified Lie bracket. We can now use the relations (E_1), (E_2) and (E_3) for f with a negative constant μ. We are looking for conditions imposed on the graph by these relations. The matrix representation of $J(Z_i)$, $1 \leq i \leq r$ is easy to obtain. If the edge Z_i as a unordered pair is equal to $X_j X_{j'}$ with $1 \leq j < j' \leq k$, then the only nonzero entries of $J(Z_i)$ are $J(Z_i)_{jj'} = -a_i$ and $J(Z_i)_{j'j} = a_i$. Hence $J^* J = 2\,\mathrm{diag}(a_1^2, \dots , a_r^2)$ and from (E_1) we have $B = \frac{1}{-(\mathrm{tr}\, f)} \mathrm{diag}(\mu - \frac{1}{2}a_1^2, \dots , \mu - \frac{1}{2}a_r^2)$. On the other hand, $\sum_{i=1}^{r} J^2(Z_i) = -\,\mathrm{diag}(d_1, \dots , d_k)$ where d_j for $1 \leq j \leq k$ is equal to $\sum_{t \in E_j} a_t^2$ with E_j denoting indexes t for which the edge Z_t has X_j as a vertex. This implies from (E_2) that $D = \frac{1}{-(\mathrm{tr}\, f)} \mathrm{diag}(\mu + \frac{1}{2}d_1, \dots , \mu + \frac{1}{2}d_k)$. As B and D are diagonal matrices, it follows from (E_3) that for an edge $Z_i = X_j X_{j'}$ we have $\mu - \frac{1}{2}a_i^2 = 2\mu + \frac{1}{2}(d_j + d_{j'})$. This means that $a_i^2 + d_j + d_{j'} = -2\mu$ whenever $Z_i = X_j X_{j'}$ is an edge.

Recall from graph theory that two distinct edges $Z_i, Z_{i'}$ of a graph (V, E) are called adjacent if they share a vertex, which will be denoted by $Z_i \sim Z_{i'}$. The line graph $L(V, E)$ is the graph whose set of vertices is E and two of them are joined if and only if they are adjacent. The adjacency matrix of a graph with vertex set $V = \{X_1, \dots , X_k\}$ is defined as the symmetric $k \times k$ matrix with a 1 in the entry jj' if $X_j X_{j'} \in E$ and zero otherwise.

Now consider the equality $a_i^2 + d_j + d_{j'} = -2\mu$ where $Z_i = X_j X_{j'} \in E$. It is clear that with the above definitions from graph theory we can rewrite this equality as the following :

$$3a_i^2 + \sum_{Z_t \sim Z_i} a_t^2 = -2\mu$$

or equivalently we have

$$(3\mathrm{Id} + \mathrm{Adj}) \begin{bmatrix} a_1^2 \\ \vdots \\ a_r^2 \end{bmatrix} = \begin{bmatrix} -2\mu \\ \vdots \\ -2\mu \end{bmatrix}$$

where Adj is the adjacency matrix of the line graph $L(V, E)$. Hence the vector

$$(3\mathrm{Id} + \mathrm{Adj})^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

has all its entries positive. This is the definition of a positive graph as introduced in [12].

Definition 3.1 *A graph is said to be positive if the vector*

$$(3\mathrm{Id} + \mathrm{Adj})^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

has all its entries positive, where Adj *is the adjacency matrix of the associated line graph.*

Hence if a two-step metric nilpotent Lie algebra associated with a graph as above is an Einstein nilradical then the graph is positive.

Conversely, suppose that the graph has no isolated vertices and is positive. Denote the vector in the above definition by $(A_1, \dots, A_r)$. We have in mind that $A_i = \frac{a_i^2}{-2\mu}$ for $1 \leq i \leq r$, where μ is the Einstein constant of the Einstein solvmanifold that will be constructed. In fact the relations that we have obtained for the matrices B and D as above, will help us to calculate explicitly μ and $\mathrm{tr}(f)$ in terms of $a_1, \dots, a_r$ and the structure of the graph. So we will have the exact values of $a_1, \dots, a_r$ as well. Now the two-step nilpotent Lie algebra $(\mathfrak{n}, [,])$ associated to the graph is clearly isomorphic to the vector space $\mathfrak{n}$ with the Lie bracket defined by $[X_j, X_{j'}] = a_i Z_i$ if $Z_i = X_j X_{j'}$ is an edge of the graph with $1 \leq j < j' \leq k$, $1 \leq i \leq r$ and zero otherwise. Let $\langle , \rangle$ be the inner product on $\mathfrak{n}$ relative to which the basis $\{X_1, \dots, X_k, Z_1, \dots, Z_r\}$ is orthonormal. We can now construct an Einstein metric solvable extension using the relations (E_1), (E_2) and (E_3). We omit the explicit formulas for the matrices B and D. So we have the following result.

Theorem 3.2 *Suppose that $\mathfrak{n}$ is a two-step nilpotent Lie algebra associated with a graph endowed with an inner product for which the vertices and the edges of the graph form an orthogonal basis. Then there exists a rank-one Einstein metric solvable extension of $\mathfrak{n}$ if and only if the graph is positive.*

Note that the case $a_1 = \cdots = a_r = 1$ leads to the regular line graph (i.e. all the vertices of the line graph have the same degree). It can be shown that this is equivalent to the case where each connected component of the graph is regular or a bipartite graph such that all the vertices in each partite set have the same degree. This was the main result in [6].

ACKNOWLEDGEMENTS. The author is indebted to the Research Council of Sharif University of Technology for support.

References

[1] D.V. Alekseevskii, Classification of quaternionic spaces with a transitive solvable group of motions, *Math. USSR-Izv.* **9** (1975), 297-339.

[2] E. Damek and F. Ricci, Harmonic analysis on solvable extensions of H-type groups, *J. Geom. Anal.* **2** (1992), 213-248.

[3] S.G. Dani and M.G. Mainkar, Anosov automorphisms on compact nil-manifolds associated with graphs, *Trans. A.M.S.* **357** (2005), 2235-2251.

[4] E. Deloff, *Naturally reductive metrics and metrics with volume preserving geodesic symmetries on NC algebras*, Dissertation, Rutgers, 1979.

[5] H.-R. Fanaï, Trois propriétés du flot géodésique en courbure négative, *C. R. Acad. Sci. Paris Sér. I Math.* **323** (1996), 1039-1045.

[6] H.-R. Fanaï, Einstein solvmanifolds and graphs, *C. R. Acad. Sci. Paris Sér. I* **344** (2007), 37-39.

[7] C.S. Gordon and M.M. Kerr, New homogeneous Einstein metrics of negative Ricci curvature, *Ann. Global Anal. Geom.* **19** (2001), 75-101.

[8] J. Heber, Homogeneous spaces of nonpositive curvature and their geodesic flow, *Internat. J. Math.* **6** (1995), 279-296.

[9] J. Heber, Noncompact homogeneous Einstein spaces, *Invent. Math.* **133** (1998), 279-352.

[10] D. Kass-Hengesch, *Exemples de variétés homogènes d'Einstein à courbure scalaire négative*, Thèse de Doctorat, Université Henri Poincaré-Nancy I, France, 1996.

[11] M. Lanzendorf, Einstein metrics with nonpositive sectional curvature on extensions of Lie algebra of Heisenberg type, *Geom. Dedicata* **66** (1997), 187-202.

[12] J. Lauret and C. Will, Einstein solvmanifolds: existence and non-existence questions, preprint.

[13] T. Wolter, Einstein metrics on solvable groups, *Math. Z.* **206** (1991), 457-471.

Received: February, 2009