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# Fixed Point Result in Probabilistic Space

**Vanita Ben Dhagat, Satyendra Nath and Nirmal Kumar**

Jai Narain college of Technology, Bairasia Road, Bhopal M.P., India
vanita1_dhagat@yahoo.co.in

**Abstract**

In this paper we prove minimization theorem in the generating space of quasi probabilistic metric space. Also we prove common fixed point theorem for the space which satisfy the minimization theorem.

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**Keywords:** Generating space of quasi probabilistic metric family, quasi compatible mapping, Minimization theorem, common fixed point

## Introduction and Preliminaries

Minimization theorem in generating space of quasi metric space was given in [4] .Let X be a non empty set and $\{d_\alpha: \alpha \in (0,1]\}$ be family of mapping $d_\alpha$ from XxX into $R^+$. $\{X, d_\alpha\}$ is called generating space of metric family if it satisfy following axioms:

(GM 1) – For any two distinct points x and y in X such that $d_\alpha(x,y) \neq 0$ ,$\forall\ \alpha \in (0,1]$

(GM 2) - $d_\alpha(x,y) = 0$ if $x=y$ and $\alpha \in (0,1]$

(GM 3) - $d_\alpha(x,y) = d_\alpha(y,x)$ For all x, y in X and $\alpha \in (0,1]$

(GM 4) - For any $\alpha \in (0,1]$ there exists $\alpha_1,\alpha_2 \in (0,\alpha]$ such that $\alpha_1+\alpha_2 \leq \alpha$ and so

$$d_\alpha(x,y) \leq d_{\alpha 1}(x,z) + d_{\alpha 2}(z,y)$$

(GM 5) - $d_\alpha(x,y)$ is non increasing and left continuous in $\alpha$ and $\forall$ x,y,z in X.

Throughout this paper, we assume that $k:(0,1] \to (0,\infty)$ is a non decreasing function satisfying the condition $K = \sup_{\alpha} k(\alpha)$

Probabilistic metric space was first introduced by Menger [1] . Later many authors Schweizer and Sklar [3] and Mishara [2] and others.

Throughout this paper D is the set of all left continuous distribution functions. A function $\Delta:[0,1]x[0,1]\to D$ is called a t-norm if the following conditions are satisfied:

(T-1) $\Delta(a,b) = \Delta(b,a)$,

(T-2) $\Delta(a,1) = a$,

(T-3) $\Delta(a,\Delta(b,c)) = \Delta(\Delta(a,b),c)$,

(T-4) $\Delta(a,b)\leq\Delta(c,d)$ for $a\leq c$ and $b\leq d$.

**Definition 1.1** A triple $(X,F,\Delta)$ is called a Menger probabilistic metric space if X is a nonempty set, $\Delta$ is a t-norm and $F:XxX\to D$ is a mapping satisfying the following conditions:

(PM-1) $F_{x,y}(t) = 1$ for all $t>0$ if and only if $x=y$,

(PM-2) $F_{x,y}(0) = 0$,

(PM-3) $F_{x,y} = F_{y,x}$,

(PM-4) $F_{x,y}(s+t) \geq \Delta(F_{x,y}(s), F_{x,y}(t))$ for all $x,y,z \in X$, $s,t \geq 0$.

**Remark:** If $\Delta$ satisfies the condition $\sup\Delta(t,t) = 1$ , then there exists topology $T$ on X such that $(X,T)$ is a Hausdorff topological space and the family of sets

$U(p) = \{U_p(\varepsilon,\lambda):\varepsilon>0,\lambda\in (0,1]\}$, $p\in X$, is a basis of neighborhood of the point p for $T$, where $U_p(\varepsilon,\lambda) = \{ x\in X: F_{x,p}(\varepsilon) >1-\lambda\}$.

Usually, the topology $T$ is called $(\varepsilon,\lambda)$ topology on $(X,F,\Delta)$.

**Proposition 1:** Let $(X,F,\Delta)$ be a Menger probabilistic metric space with the t-norm $\Delta$ satisfying the condition:

$$\sup_{t>1}\Delta(t,t) = 1 \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

For any $\alpha\in(0,1]$, we define $d_\alpha:XxX\to R^+$ as follows:

$$d_\alpha(x,y) = \inf \{ t>0: F_{x,y}(t) >1-\alpha\} \ldots\ldots\ldots\ldots\ldots (2)$$

Then (i) $(X, d_\alpha: \alpha\in(0,1])$ is a generating space of quasi metric family and

(ii) the topology $T(d_\alpha)$ on $(X, d_\alpha: \alpha\in(0,1])$ coincide with the $(\varepsilon,\lambda)$ topology $T$ on $(X,F,\Delta)$.

**Proof : (i)** From the definition of $(d_\alpha: \alpha\in(0,1])$, it is easy to see that $d_\alpha: \alpha\in(0,1])$ satisfies the condition (QM-1) and (QM-2) in definition 1 . Besides, it follows clearly that $(d_\alpha)$ is non-increasing in $\alpha$.

Next we prove that $d_\alpha$ is left continuous in $\alpha$. For any given $\alpha_1\in(0,1]$ and $\varepsilon>0$, from definition $d_\alpha$, there exists a $t_1>0$ such that $t_1< d_\alpha(x,y) + \varepsilon$ and $F_{x,y}(t_1) >1-\alpha_1$. Letting $\delta = F_{x,y}(t_1) - (1-\alpha_1) >0$ and $\lambda\in(\alpha_1-\delta,\alpha_1]$, we have

$1-\alpha_1< 1-\lambda < 1-(\alpha_1-\delta) = F_{x,y}(t_1)$, which implies that $t_1\in\{t>0: F_{x,y}(t) >1-\lambda\}$. Hence we have $d_{\alpha 1}(x,y) \leq d_\lambda(x,y) = \inf \{t>0: F_{x,y}(t) >1-\lambda\}$

$$\leq t_1 < d_{\alpha 1}(x,y)+\varepsilon,$$

which shows that $d_\alpha$ is left continuous in $\alpha$.

Now, we prove that $(X, d_\alpha: \alpha\in(0,1])$ also satisfies the condition (QM-3).
By the condition (1), for any given $\alpha\in(0,1]$, there exists an $\mu\in(0,\alpha]$ such that
$\Delta(1-\mu, 1-\mu) > 1-\alpha$
Letting $d_\mu(x,z) = \sigma$ and $d_\mu(z,y) = \beta$, from (2), for any given $\varepsilon>0$, we have
$F_{x,z}(\sigma+\varepsilon) > 1-\mu$, $F_{z,y}(\beta+\varepsilon) > 1-\mu$
and so $F_{x,y}(\sigma + \beta + 2\varepsilon) \geq \Delta( F_{x,z}(\sigma+\varepsilon), F_{z,y}(\beta+\varepsilon))$

$$\geq \Delta(1-\mu, 1-\mu) > 1-\alpha.$$

Hence we have $d_\alpha(x,y) \leq (\sigma + \beta + 2\varepsilon) = d_\mu(x,z) + d_\mu(z,y) + 2\varepsilon$.

$$\Rightarrow d_\alpha(x,y) \leq d_\mu(x,z) + d_\mu(z,y).$$

(ii) to prove the condition, it is enough to prove that for any $\varepsilon>0$ and $\alpha\in(0,1]$, $d_\alpha(x,y) < \varepsilon$ if and only if $F_{x,y}(\varepsilon) > 1-\alpha$. In fact if $d_\alpha(x,y) < \varepsilon$, from (2) , we have $F_{x,y}(\varepsilon-\mu) > 1-\alpha$. Conversely, if $F_{x,y}(\varepsilon) > 1-\alpha$, since $F_{x,y}$ is a left continuous distribution function, there exists an $\mu > 0$ such that $F_{x,y}(\varepsilon-\mu) > 1-\alpha$.
Hence $d_\alpha(x,y) \leq \varepsilon-\mu < \varepsilon$.
This complete the proof.

**Remark:** Such Menger probabilistic metric space with (2) is called generating space of quasi probabilistic metric space.

**Definition :** Let S and T be mappings from a generating space of quasi metric family $(X, d_\alpha: \alpha\in(0,1])$ into itself. The mappings S and T are said to be quasi compatible if $d_\alpha(STx_n, TSx_n) \to 0$ as $n \to \infty$ for all $\alpha\in(0,1]$. Whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n = \lim Tx_n = p$, for some p in X.

**Theorem 1**. Let $(X_1,F_1,\Delta_1)$ and $(X_2,F_2,\Delta_2)$ be two complete generating space of quasi probabilistic metric space. with t-norm satisfying the condition (1) . Let $f: X_1\to X_2$ be a mapping, $T:X_1\to X_1$ be a continuous mapping satisfying
$\inf\{t>0:F_{1\,Tx,Ty}(t) > 1-\alpha\} \leq \inf\{t>0:1/2(F_{1\,x,Ty}(t)+ F_{1\,Tx,y}(t)) > 1-\alpha\}$ and
$\inf\{t>0:F_{2\,f(Tx),f(Ty)}(t) > 1-\alpha\} \leq \inf\{t>0:1/2(F_{2\,x,Ty}(t)+ F_{2\,Tx,y}(t))> 1-\alpha\}$
for all x,y in $X_1$ and $\alpha\in(0,1]$. Let $\psi: R\to R$ be a non decreasing continuous function, bounded from below and $\phi:f(X_1)\to R$ a lower semi continuous function, bounded from below. Assume that for any $p\in X_1$ with $\inf_{x\in X_1} \psi(\phi(f(x))) < \psi(\phi(f(p)))$, there exists $q \in X_1$ with $p\neq Tq$ and

$$\text{Max}\{1/2[\inf\{t>0:(F_{1\,q,Tp}(t)> 1-\alpha\}+ \inf\{t>0:(F_{1\,p,Tq}(t)> 1-\alpha\}],$$
$$c/2[[\inf\{t>0:(F_{2\,f(q),f(Tp)}(t)> 1-\alpha\}+ \inf\{t>0:(F_{2\,f(p),f(Tq)}(t)> 1-\alpha\}]\}$$
$$\leq k(\alpha) [\psi(\phi(f(p))) - \psi(\phi(f(q)))],$$

where $c >0$ is a given constant. Then there exists $x_0$ in $X_1$, such that $\inf_{x\in X_1} \psi(\phi(f(x))) = \psi(\phi(f(x_0)))$

**Proof:** Let us suppose $\inf_{x\in X_1} \psi(\phi(f(x))) = \psi(\phi(f(y)))$ for every $y\in X_1$ and chose $p \in X_1$

with $\psi(\phi(f(p)))<\infty$. Then we define a sequence $\{p_n\}\subset X_1$ with $p_1 = p$ . Suppose $p_n$ is known and consider

$$Wn = \{w\in X_1: \max \{1/2[\inf\{t>0:(F_{1\, w,Tpn}(t)> 1-\alpha\}+ \inf\{t>0:(F_{1\, Tw,pn}(t)> 1-\alpha\}], c/2[[\inf\{t>0:(F_{2\, f(w),f(T(p_n))}(t)> 1-\alpha\}+ \inf\{t>0:(F_{2\, f(Tw), f(p_n)}(t)> 1-\alpha\}]\} \ldots\ldots..(4)$$

$\leq k(\alpha)\, [\psi(\phi(f(p_n))) - \psi(\phi(f(w)))]$ for any $\alpha\in(0,1]$. Since Wn is nonvoid set therefore there exists $w\in$ Wn such that $p_n \neq Tw$. We can choose $p_{n+1} \in$ Wn such that $p_n \neq Tp_{n+1}$ and

$$\psi(\phi(f(p_{n+1}))) \leq \inf_{x\in X_1} \psi(\phi(f(x))) - 1/2\, [[\psi(\phi(f(p_n))) - \inf_{x\in X_1} \psi(\phi(f(x)))]\ldots\ldots\ldots(5)$$

we observe that $[\psi(\phi(f(p_{n+1})))]$ is a non increasing lower bounded sequence, hence it is convergent sequence.

Now we prove that $\{p_n\}$ and $\{ f(p_n)\}$are cauchy sequences,

$$\text{Max}[\inf\{t>0:(F_{1\, T(p_n),T(p_{n+1})}(t)> 1-\alpha\}, c \inf\{t>0:(F_{2\, f(T(p_n)), f(T(p_{n+1}))}(t)> 1-\alpha\}]$$

$$\leq \max \{ 1/2[\inf\{t>0:(F_{1\, p_n ,T(p_{n+1})} (t)> 1-\alpha\}+ \inf\{t>0:(F_{1\, p_{n+1},T(p_n)}(t)> 1-\alpha\}], c/2[[\inf\{t>0:(F_{2\, f(p_n),f(T(p_{n+1}))}(t)> 1-\alpha\}+ \inf\{t>0:(F_{2\, f(T(p_n)), f(p_n)}(t)> 1-\alpha\}]\}$$

$$\leq k(\alpha)\, [\psi(\phi(f(p_n))) - \psi(\phi(f(p_{n+1})))]$$

Now for all $n,m \in N$, $n< m \Rightarrow$ there exists $\mu\leq\alpha$, $\mu = \mu(n,m)$ such that

$$\text{Max}[\inf\{t>0:(F_{1\, T(p_n),T(p_m)}(t)> 1-\alpha\}, c \inf\{t>0:(F_{2\, f(T(p_n)), f(T(p_m))}(t)> 1-\alpha\}]$$

$$\leq \sum_{j=n}^{m-1} \max \{ 1/2[\inf\{t>0:(F_{1\, p_j ,T(p_{j+1})} (t)> 1-\mu\}+ \inf\{t>0:(F_{1\, p_{j+1},T(p_j)}(t)> 1-\mu\}], c/2[[\inf\{t>0:(F_{2\, f(p_j),f(T(p_{j+1}))}(t)> 1-\mu\}+ \inf\{t>0:(F_{2\, f(T(p_j)), f(p_{j+1})}(t)> 1-\mu\}]\}$$

Hence for all $n,m \in N$, $n< m$:

$$\text{Max}[\inf\{t>0:(F_{1\, T(p_n),T(p_m)}(t)> 1-\alpha\}, c \inf\{t>0:(F_{2\, f(T(p_n)), f(T(p_m))}(t)> 1-\alpha\}]$$

$$\leq k(\mu) \sum_{j=n}^{m-1} [\psi(\phi(f(p_j))) - \psi(\phi(f(p_{j+1})))]$$

$$\leq k(\alpha)\, [\psi(\phi(f(p_n))) - \psi(\phi(f(p_m)))]$$

for some $\alpha$ with $0< \alpha_{j+1} \leq \alpha_k \leq \alpha$, $j = n, \ldots\ldots, m-1$ and

$$\inf \{t>0: F_{1\, p_n,p_{n+1}} (t) >1-\alpha\} \leq \inf\{t>0:(F_{1\, p_n,T(p_{n+1})}(t)> 1-\mu\} + \inf\{t>0:(F_{1\, T(p_n),T(p_{n+1})}(t)> 1-\mu\} + \inf\{t>0:(F_{1\, T(p_n),p_{n+1}}(t)> 1-\mu$$

$$\leq \{ 1/2[\inf\{t>0:(F_{1\, p_n ,T(p_{n+1})} (t)> 1-\mu\}+ \inf\{t>0:(F_{1\, p_{n+1},T(p_n)}(t)> 1-\mu\}]$$

$$\leq 3\, k(\alpha)\, [\psi(\phi(f(p_n))) - \psi(\phi( f(p_{n+1})))]$$

Then , for $n<m$

$$\inf\{t>0:(F_{1\, T(p_n),T(p_m)}(t)> 1-\alpha\}\leq 3\, k(\alpha)\, [\psi(\phi(f(p_n))) - \psi(\phi( f(p_m)))]$$

Similarly,

$$\inf\{t>0:(F_{2\, f(T(p_n)), f(T(p_{n+1}))}(t)> 1-\alpha\} \leq 3\, k(\alpha)\, [\psi(\phi(f(p_n))) - \psi(\phi( f(p_m)))]$$

Hence $\{p_n\}$ and $\{ f(p_n)\}$are cauchy sequences.

Let $\lim p_n = u$ and $\lim f(p_n) = v$ as $n\to\infty$.

Since $f$ is closed therefore $f(u) = v \in f(X_1)$.

By the continuity of $\psi$ and lower semi continuity of $\phi$, we have

$\psi(\phi(v)) \leq \lim \psi(\phi(f(p_n))) = \lim \psi(\phi(f(p_{n+1})))$
Let $\inf \psi(\phi(f(x))) = \lambda_0 \in R$ . From $\psi(\phi(f(p_{n+1}))) \leq \lambda_0 + 1/2 \{\psi(\phi(f(p_n))) - \lambda_0\}$, we have $\lim \psi(\phi(f(p_{n+1}))) \leq \lambda_0 /2 + 1/2 \lim \psi(\phi(f(p_n))) = \lambda_0 /2 + 1/2 \lim \psi(\phi(f(p_{n+1})))$.
Finally, $\psi(\phi(f(u))) = \psi(\phi(v)) \leq \lim \psi(\phi(f(p_{n+1}))) \leq \lambda_0 = \inf \psi(\phi(f(x))) \leq \psi(\phi(f(u)))$ which is contradiction. Therefore there exists an $x_0 \in X$ such that
$\inf \psi(\phi(f(x))) = \psi(\phi(f(x_0)))$.

**Theorem 2 :** Let $(X_1,F_1,\Delta_1)$ and $(X_2,F_2,\Delta_2)$ be two complete Menger probabilistic metric space with t-norm satisfying the condition (1) . Let $f : X_1\rightarrow X_2$ be a mapping, $T:X_1\rightarrow X_1$ be a continuous mapping satisfying
$\inf\{t>0:F_{1\, Tx,Ty}(t) > 1-\alpha\} \leq \inf\{t>0:1/2(F_{1\, x,Ty}(t)+ F_{1\, Tx,y}(t)) > 1-\alpha\}$ and
$\inf\{t>0:F_{2\, f(Tx)\, ,\, f(Ty)}(t) > 1-\alpha\} \leq \inf\{t>0:1/2(F_{2\, x,Ty}(t)+ F_{2\, Tx,y}(t))> 1-\alpha\}$
for all x,y in $X_1$ and $\alpha\in(0,1]$. Let $\psi$: $R\rightarrow R$ be a non decreasing continuous function,

bounded from below and $\phi:f(X_1)\rightarrow R$ a lower semi continuous function, bounded from

below . Let $S :X_1\rightarrow X_1$ be continuous mapping. Further, if S and T are quasi compatible and $\text{Max}\{1/2[\inf\{t>0:(F_{1\, x,TSx}(t)> 1-\alpha\}+ \inf\{t>0:(F_{1\, Tx,Sx}(t)> 1-\alpha\}],$
$c/2[[\inf\{t>0:(F_{2\, f(x),f(TSx)}(t)> 1-\alpha\}+ \inf\{t>0:(F_{2\, f(Tx), f(Sxs)}(t)> 1-\alpha\}]\}$
$\leq k(\alpha) [\psi(\phi(f(x))) - \psi(\phi(f(Sx)))]$, for every x and $\alpha\in(0,1]$ ,where c >0 is a given constant, then there exists unique common fixed point.

**Proof:** By the **theorem 1**, we have $\inf \psi(\phi(f(x))) = \psi(\phi(f(x_0)))$.
Suppose $x_0 \neq TSx_0$ or $Sx_0 \neq Tx_0$. Then for some $\alpha\in(0,1]$
$0< 1/2[\inf\{t>0:(F_{1x0,TSx0}(t)> 1-\alpha\}+ \inf\{t>0:(F_{1\, Tx0,Sx0}(t)> 1-\alpha\}]$
$\leq k(\alpha) [\psi(\phi(f(x_0))) - \psi(\phi(f(Sx_0)))]\leq 0$
which is the contradiction. Therefore $x_0 = TSx_0$ or $Sx_0 = Tx_0$ and also $x_0 = Tx_0$ .
Moreover for every $\alpha\in(0,1]$ there exists $\mu \in(0, \alpha]$ such that

$$\begin{aligned}\inf\{t>0:(F_{1x0,Tx0}(t)> 1-\alpha\} &\leq \inf\{t>0:(F_{1x0,TSx0}(t)> 1-\mu\}+ \inf\{t>0:(F_{1\, TSx0,Tx0}(t)> 1-\mu\}]\\ &= \inf\{t>0:(F_{1\, TSx0,Tx0}(t)> 1-\mu\}]\\ &\leq 1/2[\inf\{t>0:(F_{1Sx0,Tx0}(t)> 1-\mu\}+ \inf\{t>0:(F_{1\, x0,TSx0}(t)> 1-\mu\}]] = 0.\end{aligned}$$

Thus $x_0 = Tx_0$ ( = $Sx_0$ )
Uniqueness can obtain by the given condition.

**Corollary::** Let $(X_1,F_1,\Delta_1)$ and $(X_2,F_2,\Delta_2)$ be two complete Menger probabilistic metric space with t-norm satisfying the condition (1) .Let $f : X_1\rightarrow X_2$ be a closed mapping, and $\phi:f(X_1)\rightarrow R$ a lower semi continuous function, bounded from below . Let $S :X_1\rightarrow X_1$ be a mapping such that
$\max\{\inf\{t>0:(F_{1S\, x,x}(t)>1-\alpha\}, c \inf\{t>0:(F_{2\, f(S(x)),f(x)}(t)>1-\alpha\}\}$
$\leq k(\alpha) [\phi(f(x)) - \phi(f(S(x)))$
for every x and $\alpha\in(0,1]$ ,where c >0 is a given constant, then there exists unique common fixed point.

**Proof:** On taking T = I and $\psi$ = I in theorem 2.

## References

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