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Summation Formulae and Stirling Numbers

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Abstract

We exploit methods of operational and combinatorial nature to get a class of summation formulae involving special functions and polynomials. The results obtained in this paper complete and integrate previous investigations obtained with different methods.

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1 INTRODUCTION

It has been recently demonstrated [4] that a class of summation formulae, often occurring in pure mathematics, in applied mathematics and in physics, can be

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expressed in terms of infinite variable Hermite polynomials [2]. The procedure developed in [4] has benefited from the use of operational methods and has provided a common framework to deal with summation formulae like

$$\begin{array}{l} A(x|m) = \sum\limits_{n=-\infty}^{\infty} n^m J_n(x), \\ B(x|m) = \sum\limits_{n=0}^{\infty} \frac{n^m}{n!} H_n(x) \end{array} , \tag{1}$$

which are frequently exploited to derive distribution moments in problems, concerning e.g. radiation emission by charged particles or quantum harmonic oscillators. In the eq. (1) $J_n(x)$ – the cylindrical Bessel functions of the first kind and $H_n(x)$ – the Hermite polynomials [1]. Although the use of the infinite variable Hermite polynomials offers a very flexible tool to treat this type of problems, we will reconsider the topics treated in [4] from different point of view, involving operational methods and the properties of the Stirling numbers. To this aim we recall few notions, which will provide the back-bone of the discussion, developed in the following sections.

The Stirling numbers of second kind are defined as follows:

$$\begin{array}{l} S_2(m,k) = \frac{1}{k!} \sum\limits_{r=0}^{k-1} (-1)^r \left(\begin{array}{c} k \\ r \end{array} \right) (k-r)^m, \\ S_2(m,0) = \delta_{m,0} \end{array} \tag{2}$$

they have been introduced in combinatorial analysis and yield the number of possibilities in which a set of $n$elements can be partitioned in $m$non empty sets [1]

The operator

$$\hat{E}_x = x\hat{D}_x, \hat{D}_x = \frac{d}{dx} \quad , \tag{3}$$

sometimes called Euler dilatation operator, is often exploited in classical and quantum optics to treat charged beam transport problems [7]. Its *m-th* power can be written in terms of Stirling numbers of the second kind as follows [8]:

$$\hat{E}_x^m = \sum_{k=1}^{m} S_2(m,k) x^k \hat{D}_x^k. \tag{4}$$

We will use the properties of the numbers$S_2(m,k)$ and of the operator $\hat{E}_m$ to derive new summation formulae and demonstrate how existing summation formulae can be obtained in a different framework.

The first example we consider is associated with the case of the geometric series. From the formula (4) we can obtain the summation formulae, involving this series as an example:

$$G_m(x)=\sum_{n=0}^{\infty} n^m x^n,\ |x|<1 \quad , \tag{5}$$

by proceeding as follows:

$$\sum_{n=0}^{\infty} n^m x^n=\hat{E}_x^m \sum_{n=0}^{\infty} x^n=\hat{E}_x^m\left(\tfrac{1}{1-x}\right)=\sum_{k=1}^{m} S_2(m,k)\left(\tfrac{k!x^k}{(1-x)^{k+1}}\right), \quad . \tag{6}$$
$$|x|<1$$

In analogous way we get the expression for the finite sum, involving Stirling numbers:

$$\begin{aligned}&\sum_{n=0}^{r} n^m x^n=\hat{E}_x^m \sum_{n=0}^{r} x^n=\hat{E}_x^m\left(\tfrac{x^{r+1}-1}{x-1}\right)=\\ &=-\sum_{k=1}^{m} S_2(m,k)x^k \sum_{s=0}^{k}\left(\begin{array}{c}k\\ s\end{array}\right)\left[\left(\tfrac{(r+1)!}{(r+1-s)!}x^{r+1-s}-\delta_{s,0}\right)\tfrac{(k-s)!}{(1-x)^{k-s+1}}\right]\end{aligned} \tag{7}$$

A less trivial application concerns the evaluation of the moments of the Poisson distribution, namely

$$\langle n^m\rangle=\sum_{n=0}^{\infty}\frac{n^m}{n!}x^n\exp(-x). \tag{8}$$

The use of the same procedure as given above yields:

$$\langle n^m\rangle=\exp(-x)\hat{E}_x^m\sum_{n=0}^{\infty}\frac{1}{n!}x^n=\sum_{k=1}^{m} S_2(m,k)x^k,$$

which is recognized as the well known Dobinsky identity [10][2] .

[2]Also note that the following equality for the series:
$\sum_{n=0}^{\infty}\frac{(a+bn)^m}{n!}=e\sum_{s=0}^{m}\left(\begin{array}{c}m\\ s\end{array}\right)a^{m-s}b^s\sum_{r=1}^{s}S_2(s,r),$
which in can be written in the following umbral form:
$\sum_{n=0}^{\infty}\frac{(a+bn)^m}{n!}=e(a+(B))^m,(B)^p=b^p\sum_{r=1}^{p}S_2(p,r).$

It is evident that by combining the properties of the Stirling numbers with the methods of operational nature and with the properties of special functions one can get a very flexible tool for the derivation of a plethora of summation formulae, which can be exploited in different applicative fields.

Indeed, let us consider the following series:

$$\begin{array}{l}\sum\limits_{n=0}^{\infty}\frac{n^m}{(n!)^2}(-1)^n x^n=\hat{E}_x^m\sum\limits_{n=0}^{\infty}\frac{(-1)^n x^n}{(n!)^2}=\hat{E}_x^m C_0(x),\\ C_m(x)=\sum\limits_{r=0}^{\infty}\frac{(-1)^r x^r}{r!(r+m)!}\end{array} \tag{9}$$

with $C_m(x)=x^{-\frac{m}{2}}J_m(2\sqrt{x})$ being the Tricomi Bessel function of the first kind and satisfying the following recurrence [5]:

$$\frac{d}{dx}C_m(x)=-C_{m+1}(x). \tag{10}$$

With the help of the above recurrence relations we obtain:

$$\sum_{n=0}^{\infty}\frac{n^m}{(n!)^2}(-x)^n=\sum_{k=1}^{m}(-1)^k S_2(m,k)x^k C_k(x)=\sum_{k=1}^{m}(-1)^k S_2(m,k)x^{\frac{k}{2}}J_k(2\sqrt{x}), \tag{11}$$

which provides a kind of generalization of the Dobinsky identity (see also ref. [4] for further comments). The use of the multi-index Tricomi functions yields further generalizations:

$$\begin{array}{l}\sum\limits_{n=0}^{\infty}\frac{n^m}{(n!)^{q+1}}(-x)^n=\sum\limits_{k=1}^{m}(-1)^k S_2(m,k)x^k C_{\{k\}}(x),\\ C_{\{n_s\}}(x)=\sum\limits_{r=0}^{\infty}\frac{(-1)^r x^r}{r!\prod\limits_{s=1}^{q}(r+n_s)!}\end{array}, \tag{12}$$

where $C_{\{n_s\}}(x)$ – the function with q indices [5], satisfying the recurrence

$$\frac{d}{dx}C_{\{n_s\}}(x)=-C_{\{n_s+1\}}(x). \tag{13}$$

The examples and remarks given in this introduction are sufficient for the purposes of this paper. In the next sections we will extend the outlined method to different families of special functions and polynomials.

2 SUMMATION FORMULAE OF SPECIAL POLYNOMIALS

In this section we will show how the method can be exploited to derive the closed form of the summation, involving classical and generalized polynomials.

We start by considering a class of polynomials, currently known as the Appèl family, which generating function is specified by [14]:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} a_n(x) = A(t) \exp(xt), \tag{14}$$

where $A(t)$ is a continuous function with at least one derivative. The properties of the $a_n(x)$ crucially depend on the form of $A(t)$, but if we take the derivative of both sides of (14) with respect to x, we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \hat{D}_x a_n(x) = tA(t) \exp(xt) = \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} a_n(x), \tag{15}$$

which, after equating coefficients for$t$ powers in the right hand side and the left hand side, yields the well known recurrence:

$$\hat{D}_x a_n(x) = n a_{n-1}(x). \tag{16}$$

In view of this identity, we also find that eq. (14) can also be cast in the form

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} a_n(x) = A(\hat{D}_x) \exp(xt) = A(\hat{D}_x) \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n, \tag{17}$$

which allows the important conclusion that the Appèl polynomials can be defined by means of the Appèl operator $A(\hat{D}_x)$ as follows

$$a_n(x) = A\left(\hat{D}_x\right) x^n. \tag{18}$$

Hence the summation formula

$$\eta_m(x|A) = \sum_{n=0}^{\infty} \frac{n^m}{n!} a_n(x) \tag{19}$$

can be written in the following form:

$$\begin{aligned} \eta_m(x|A) &= A\left(\hat{D}_x\right)\, \eta_m(x|1) \\ \eta_m(x|1) &= \sum_{k=1}^{m} S_2(m,k)\, x^k \exp(x) \end{aligned} . \tag{20}$$

The explicit form of the summation depends on the form of the Appèl operator.

The Hermite polynomials in two variables [2]

$$H_n(x,y) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2r} y^r}{(n-2r)!r!} \tag{21}$$

belong to the Appél family. They are generated by eq. (14), where

$$A(t) = \exp(yt^2) \tag{22}$$

and they reduce to the ordinary Hermite polynomials in case of the specific choice of the variables $\left(H_n(2x,-1) = H_n(x), H_n(x,-\frac{1}{2}) = He_n(x)\right)$. Being the relevant Appèl operator given by

$$A(\hat{D}_x) = \exp(y\hat{D}_x^2), \tag{23}$$

we can also define the $H_n(x,y)$ through the operational identity:

$$\exp(y\hat{D}_x^2)x^n = H_n(x,y). \tag{24}$$

Thus the associated summation

$$\eta_m(x,y|H) = \sum_{n=0}^{\infty} \frac{n^m}{n!} H_n(x,y) \tag{25}$$

can be written in the following form:

$$\eta_m(x,y) = \exp(y\hat{D}_x^2)\sum_{n=0}^{\infty}\frac{n^m}{n!}x^n = \exp(y\hat{D}_x^2)\eta_m(x|1) \quad . \tag{26}$$

Here we face the problem of specifying the action of the exponential operator on the function $x^k \exp(x)$. This task can be accomplished using the methods illustrated in [4], [10] and reported in Appendix I for completeness, which yield:

$$\exp(y\hat{D}_x^2)\left[x^k \exp(x)\right] = \exp(x+y)H_k(x+2y,y). \tag{27}$$

Eventually, we obtain the following expression:

$$\sum_{n=0}^{\infty}\frac{n^m}{n!}H_n(x,y) = \exp(x+y)\sum_{k=1}^{m}S_2(m,k)H_k(x+2y,y). \tag{28}$$

Analogous result for the ordinary Hermite polynomials has been derived in [13]. However, it was obtained in the framework of a different method.

The Bernoulli polynomials $B_n(x)$ (see, for example, [1]) belong to the Appèl family with

$$A(\hat{D}_x) = \frac{\hat{D}_x}{e^{\hat{D}_x}-1} \quad , \tag{29}$$

and therefore we end with

$$\eta_m(x|B) = \sum_{n=0}^{\infty}\frac{n^m}{n!}B_n(x) = \frac{\hat{D}_x}{e^{\hat{D}_x}-1}\eta_m(x|1). \tag{30}$$

The form (30), albeit leading to the expression, which does not write in simple terms, can be exploited to prove the following differential-difference recursion

$$\eta_m(x+1|B) - \eta_m(x|B) = \eta'_m(x|1). \tag{31}$$

We will further discuss the problems underlying the action of the Appèl operator on the $\eta_m(x|1)$ in the following chapters.

Before proceeding, some remarks are necessary. We have underlined that the polynomials $H_n(x,y)$ are the two-variable polynomials, belonging to the Appèl family. This is strictly true if $x-$ the variable and $y-$ the parameter. With the different from the above identification (i.e. $y-$ the variable and $x-$ the parameter) they can be rather considered Sheffer polynomials than Appèl polynomials. It will be demonstrated and discussed in the following chapters. Although the conventional Laguerre polynomials cannot be recognized as Appèl polynomials, the proposed in this article technique can be applied for them same successfully.

Let us introduce the two variable Laguerre polynomials, defined as follows [5]:

$$L_n(x,y) = n!\sum_{r=0}^{n} \frac{(-1)^r x^r y^{n-r}}{(r!)^2 (n-r)!}. \tag{32}$$

They are characterized by the following generating function:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x,y) = \exp(yt) C_0(xt). \tag{33}$$

The formula (33) for the generating function is true only in case when x- the *variable* and y- the *parameter*; the converse will be discussed in the Appendix II. In any case we find from the eq.(33) that a slightly modified version of the previously outlined procedure yields the following identity:

$$\begin{aligned} &\sum_{n=0}^{\infty} \tfrac{n^m}{n!} L_n(x,y) = (t\tfrac{\partial}{\partial t})^m [\exp(yt) C_0(xt)]|_{t=1} \\ &= \exp(y) \sum_{k=1}^{m} S_2(m,k) F_k(x,y) \\ &F_k(x,y) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} y^{k-r} x^r C_r(x) \end{aligned} \quad . \tag{34}$$

Now let us apply the same method for the family of the Legendre polynomials, considering the following two variable polynomials [5]:

$$S_n(x,y) = n!\sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^r y^{n-2r} x^r}{(n-2r)!(r!)^2} \tag{35}$$

with the their generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} S_n(x,y) = \exp(yt) C_0(xt^2). \tag{36}$$

For these polynomials we obtain by analogy with eq. (20) the following identities:

$$\begin{array}{l} \sum\limits_{n=0}^{\infty} \frac{n^m}{n!} S_n(x,y) = (t\frac{\partial}{\partial t})^m [\exp(yt) C_0(xt^2)]|_{t=1} = \exp(y) \sum\limits_{k=1}^{m} S_2(m,k) G_k(x,y) \\ G_k(x,y) = \sum\limits_{r=0}^{k} \left(\begin{array}{c} k \\ r \end{array} \right) y^{k-r} \left(\frac{\partial}{\partial t}\right)^r C_0(xt^2)|_{t=1} = \sum\limits_{r=0}^{k} \left(\begin{array}{c} k \\ r \end{array} \right) y^{k-r} N_r(x), \\ N_r(x) = r! \sum\limits_{p=0}^{\left[\frac{r}{2}\right]} \frac{(2)^{r-2p}(-x)^{r-p}}{(r-2p)!p!} C_{r-p}(x) \end{array} . \tag{37}$$

Note that when deriving the previous expression, we used the following identity:

$$\left(\frac{d}{dt}\right)^r C_0(xt^2)|_{t=1} = N_r(x), \tag{38}$$

where the function $N_r(x)$ can be seen as a kind of discrete Hermitian convolution of the Bessel functions.

The link with the Legendre polynomials is readily obtained by recalling that [5]

$$P_n(y) = S_n\left(-\frac{1-y^2}{4}, y\right). \tag{39}$$

Before concluding this section we will prove the important identity, involving the polynomials of the truncated exponential (see the Appendix III):

$$e_p(x) = \sum_{r=0}^{p} \frac{x^r}{r!}, \frac{d}{dx} e_p(x) = e_{p-1}(x). \tag{40}$$

Indeed, let's consider the sum $\sum\limits_{n=0}^{p} \frac{n^m}{n!} x^n$, which appears in the problems, associated with physics of super-Gaussian beams. By using the methods, demonstrated previously on the above given examples, we easily obtain the following expression for the sum:

$$\sum_{n=0}^{p} \frac{n^m}{n!} x^n = \sum_{k=1}^{m} S_2(m,k) x^k e_{p-k}(x). \tag{41}$$

In the following chapters we will consider more complicated case of multi-variable polynomials.

3 SUMMATION FORMULAE AND MULTI-INDEX POLYNOMIALS

Now let's consider several summation formulae important for the applications in physics, in particular, when treating the quantum harmonic oscillator problems. These formulae involve the products of Hermite polynomials, namely

$$G_m(x,y;z,w) = \sum_{n=0}^{\infty} \frac{n^m}{n!} H_n(x,y) H_n(z,w). \tag{42}$$

The closed expressions can be obtained for these polynomials by means of the previously developed method as sketched below:

$$\begin{gathered} G_m(x,y;z,w) = \exp(y\hat{D}_x^2)\left(x\tfrac{\partial}{\partial x}\right)^m \exp(zx + wx^2) = \\ = \exp(y\hat{D}_x^2) \sum_{k=1}^{m} S_2(m,k) x^k H_k(z + 2wx, w) \exp(xz + wx^2) \end{gathered} \tag{43}$$

with the help of the following identity (see Appendix I):

$$\hat{D}_x^k \exp(ax + bx^2) = H_k(a + 2bx, b) \exp(ax + bx^2). \tag{44}$$

Eventually, let us write explicitly the action of the exponential operator in eq. (43), making use of the Gauss transform:

$$\exp(a\hat{D}_x^2) f(x) = \frac{1}{2\sqrt{\pi a}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\xi)^2}{4a}\right) f(\xi) d\xi \quad , \tag{45}$$

which yields:

$$
\begin{array}{l}
G_m(x,y;z,w)=\sum\limits_{k=0}^{m}S_2(m,k)g_k(x,y;z,w),\\
g_k(x,y;z,w)=\frac{1}{\sqrt{1-2yw}}\exp\left(-\frac{1}{4}\left[\frac{x^2}{y}-\frac{B^2}{A}\right]\right)H_{k,k}\left(\frac{B}{2A},\frac{1}{4A};\frac{wB}{A}+z,\frac{w^2}{A}+w|\frac{w}{A}\right),\\
A=\frac{1-4wy}{4y},B=\frac{x+2zy}{2y},A>0
\end{array}
\tag{46}
$$

where

$$
H_{m,n}(x,y;z,w|\tau)=m!n!\sum_{s=0}^{\min[m,n]}\frac{\tau^s H_{m-s}(x,y)H_{n-s}(w,z)}{s!(m-s)!(n-s)!}. \tag{47}
$$

Note that the two index Hermite polynomials [2] (see Appendix IV for further details of their derivation) can also be constructed using the following operational rule:

$$
H_{m,n}(x,y;z,w|\tau)=\exp(y\hat{D}_x^2+\tau\hat{D}_{x,z}^2+w\hat{D}_z^2)\,(x^m z^n)\,. \tag{48}
$$

The multi-index polynomials play central role in various problems in physics; in particular such Hermite polynomials are crucial for the study of quantum oscillators entangled states [14]. The calculation of the following summation formula

$$
\eta_{p,q}^{(2)}(x,y;z,w|\tau)=\sum_{m,n=0}^{\infty}\frac{m^p}{m!}\frac{n^q}{n!}H_{m,n}(x,y;z,w|\tau) \tag{49}
$$

can be performed with slight extension of the developed in this work formalism, following which we obtain:

$$
\begin{array}{l}
\eta_{p,q}^{(2)}(x,y;z,w|\tau)=\exp(y\hat{D}_x^2+\tau\hat{D}_{x,z}^2+w\hat{D}_z^2)[\eta_p(x)\eta_q(z)]=\\
\exp(x+y+z+w+\tau)\sum\limits_{k=1}^{p}\sum\limits_{k'=1}^{q}S_2(p,k)S_2(q,k')H_{k,k'}(x+2y+\tau,y;z+2w+\tau,w|\tau)
\end{array}.
\tag{50}
$$

The details of the calculation of the above formula are given in the Appendix IV.

4 SUMMATION FORMULAE AND MONOMIALITY

According to the monomiality treatment [2] the family of polynomials $p_n(x), n \in N$, $p_0(x) = 1$ can be considered a quasi monomial if we can identify two operators $\hat{M}, \hat{P}$, playing the role of the multiplicative and the derivative operators, so that the following equalities hold:

$$\begin{array}{l} \hat{M} p_n(x) = p_{n+1}, \\ \hat{P} p_n(x) = n p_{n-1}(x), \ . \\ p_0(x) = 1 \end{array} \tag{51}$$

It also follows from (51) that[3] the following operator identities are true:

$$\hat{M}^n = p_n(x) \tag{52}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} p_n(x) = \exp(t\hat{M}), \tag{53}$$

where $p(x)$ – the eigenfunction of the operator $\hat{M}$ [2]. It goes by itself that $\left[\hat{P}, \hat{M}\right] = \hat{1}$, and hence

$$(\hat{M}\hat{P})^m = \sum_{k=1}^{m} S_2(m,k) \hat{M}^k \hat{P}^k, \tag{54}$$

resulting in the following expression for the summation formulae, involving the p_n polynomials:

$$\sum_{n=0}^{\infty} \frac{n^m}{n!} p_n(x) = \sum_{k=1}^{m} S_2(m,k) \hat{M}^k \exp(\hat{M}). \tag{55}$$

We will now use the formalism, associated with the monomiality principle, to make some useful remarks. The Laguerre polynomials have been shown

[3]It is understood that the lhs of the first of eq. (50) should be understood as an operational identity in which the exponential operator acts on the unity (see ref. (2) for further comments)

to belong to the family of quasi-monomials and the relevant multiplicative operator is provided by [2] the following form:

$$\hat{M} = y - \hat{D}_x^{-1},$$

where $\hat{D}_x^{-1}$ – the inverse of the derivative operator [2], [2], the action of which on a given function $f(x)$ is defined as follows:

$$\hat{D}_x^{-1} f(x) = \int_0^x f(x')dx'. \tag{56}$$

According to the above rule (56) the repeated action of the inverse derivative operator on unity yields $\hat{D}_x^{-n} 1 = x^n/n!$ and on account of the fact that

$$\exp(-\hat{D}_x^{-1}) = C_0(x) \tag{57}$$

we readily obtain the result quoted in eq. (33).

In case of Hermite polynomials we can identify the multiplicative and derivative operators as follows:

$$\hat{M} = x + 2y\hat{D}_x, \hat{P} = \hat{D}_x \tag{58}$$

and, therefore, their product appears to be

$$\hat{M}\hat{P} = \hat{\theta} = x\hat{D}_x + 2y\hat{D}_x^2 \quad , \tag{59}$$

where $\hat{\theta}$ – the Hermite operator, defined in such way that

$$\hat{\theta}^m H_n(x, y) = n^m H_n(x, y). \tag{60}$$

According to the previous formulae we can write the action of the $\hat{\theta}$ operator on a generic function $f(x)$ as follows:

$$\hat{\theta}^m f(x) = \sum_{k=1}^{m} S_2(m, k) \sum_{s=0}^{k} (2y)^s \binom{k}{s} H_{k-s}(x, y) f^{(k+s)}(x), \tag{61}$$

where $f^{(k)}(x)$ – the derivative of the order k. The above equation has been derived by exploiting the Burchnall identity [11]:

$$(x + 2y\hat{D}_x)^k = \sum_{s=0}^{k} (2y)^s \begin{pmatrix} k \\ s \end{pmatrix} H_{k-s}(x, y)\hat{D}_x^s \quad . \tag{62}$$

We can therefore use the above relations to derive the result quoted in eq. (28) (see Appendix I for further comments).

Furthermore, in general we can construct the derivative and multiplicative operators for the Appèl family by noting that on account of (16) the Appèl $\hat{P}$ operator is just the ordinary derivative and thus we can make use of the recurrence

$$a_{n+1}(x) = A\left(\hat{D}_x\right)[x(x^n)] = A(\hat{D}_x)x\left[A\left(\hat{D}_x\right)\right]^{-1} a_n(x) \tag{63}$$

to obtain the following result for the operator M:

$$\hat{M} = A(\hat{D}_x)x\left[A\left(\hat{D}_x\right)\right]^{-1} = x + \left[A(\hat{D})\right]^{-1} A'(\hat{D}_x) \quad , \tag{64}$$

where the prime denotes the derivative of the function $A(t)$. Hence the Appèl polynomials are shown to satisfy the following differential equation:

$$\left[x + \frac{A'\left(\hat{D}_x\right)}{A\left(\hat{D}_x\right)}\right]\hat{D}_x a_n(x) = n a_n(x) \tag{65}$$

and we can write:

$$\eta_m(x|A) = A\left(\hat{D}_x\right)\sum_{k=1}^{m} S_2(m,k)x^k \exp(x) = \sum_{k=1}^{m} S_2(m,k)\hat{M}^k \exp(x), \tag{66}$$

where $\hat{M}$ is defined by the eq. (64).

5 SHEFFER POLYNOMIALS, STIRLING NUMBERS AND SUMMATION FORMULAE

The Sheffer polynomials are the natural extension of the Appèl polynomials; they are generated by the following generating function [11], [12]:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} s_n(x) = A(t) \exp(xB(t)) \tag{67}$$

with $B(t)$, satisfying the same properties as $A(t)$ does. We follow the point of view of ref. [14] to prove their quasi-monomiality. To this aim we take the freedom of calling the polynomials $\tilde{s}_n(x)$ Pre-Sheffer with their generating function defined as follows:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \tilde{s}_n(x) = \exp(xB(t)) \quad . \tag{68}$$

Multiplying both sides of eq.(68) by the operator $B^{-1}(\hat{D}_x)$, we obtain:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} B^{-1}(\hat{D}_x) \tilde{s}_n(x) = B^{-1}(\hat{D}_x) \exp(xB(t)) = t \exp(xB(t)), \tag{69}$$

thus finding that the derivative operator for this family of polynomials is in fact

$$\hat{P} = B^{-1}(\hat{D}_x). \tag{70}$$

The multiplicative operator can now be defined by taking the derivative with respect to $t$of both sides of eq. (68):

$$\sum_{n=0}^{\infty} \frac{nt^{n-1}}{n!} \tilde{s}_n(x) = xB'(t) \exp(xB(t)) = xB'(B^{-1}(\hat{D}_x)) \exp(xB(t)), \tag{71}$$

which implies that the relevant multiplicative operator can be cast in the following form:

$$\hat{M} = xB'(\hat{P}) \tag{72}$$

and according to the eq. (52) we also obtain:

$$\left[xB'(\hat{P})\right]^n = \tilde{s}_n(x) \tag{73}$$

with $\hat{P}$ given by the eq. (70).

Let us now consider the Sheffer polynomials and note that they can be defined through the following operational rule:

$$s_n(x) = A(\hat{P})\tilde{s}_n(x). \tag{74}$$

It is now clear that the Sheffer polynomials are just the corresponding Appèl polynomials on the space of the Pre-Sheffer polynomials with the following identification:

$$\begin{array}{l} x^n \leftrightarrow \tilde{s}_n(x), \\ \hat{D}_x \leftrightarrow B^{-1}(\hat{D}_x) \end{array} . \tag{75}$$

Thus, we obtain for the Sheffer polynomials the following realization of the Weyl-Heisenberg operators:

$$\begin{array}{l} \hat{P} = B^{-1}(D_x), \\ \hat{M} = (x + \frac{A'(\hat{P})}{A(\hat{P})})B'(\hat{P}) \end{array} . \tag{76}$$

The fact that we have obtained the realization of the Weyl-Heisenberg algebra generators in the form (76) opens new perspectives on the possibility of getting new summation formulae involving Sheffer polynomials.

In case of the binomial polynomials $b_n(x)$ with the generating function $e^{(\exp(t)-1)x}$, we have $B(t) = \exp(t) - 1$ and therefore the operators P and M become

$$\begin{array}{l} \hat{P} = \ln(1 + \hat{D}_x), \\ \hat{M} = x(1 + \hat{D}_x) \end{array} . \tag{77}$$

We also obtain the following summation formulae, involving the Stirling numbers:

$$\begin{array}{l}\sum_{n=0}^{\infty}\frac{n^m}{n!}b_n(x)=\sum_{k=1}^{m}S_2(m,k)\left[x(1+\hat{D}_x)\right]^k\exp(x(e-1))=\\ \sum_{k=1}^{m}S_2(m,k)\left[(1+\hat{D}_x)x-1\right]^k\exp(x(e-1))=\\ =\sum_{k=1}^{m}S_2(m,k)\sum_{s=0}^{k}\left(\begin{array}{c}k\\ s\end{array}\right)(-1)^{k-s}\left[(1+\hat{D}_x)x\right]^s\exp(x(e-1))=\\ \sum_{k=1}^{m}S_2(m,k)\sum_{s=0}^{k}\left(\begin{array}{c}k\\ s\end{array}\right)(-1)^{k-s}\sum_{r=1}^{s+1}S_2(s+1,r)x^{r-1}\times\\ .(1+\hat{D}_x)^{r-1}\exp(x(e-1))=\\ \sum_{k=1}^{m}S_2(m,k)\sum_{s=0}^{k}\left(\begin{array}{c}k\\ s\end{array}\right)(-1)^{k-s}\sum_{r=1}^{s+1}S_2(s+1,r)(ex)^{r-1}\exp(x(e-1))\end{array}. \quad (78)$$

Other examples will be discussed in the forthcoming publications.

6 CONCLUDING REMARKS

In this section we will consider three other examples in order to underline the versatility of the general approach developed in this article. Despite the first example is rather academic, it demonstrates that closed expressions can be obtained either in apparently complicated cases.

Let us consider the summation $\sum_{n=0}^{\infty}\frac{J_m(nx)}{n!}$, which can be written in the following closed form by using the series expansion of the first kind Bessel function and the notation of umbral nature:

$$\begin{array}{l}\sum_{n=0}^{\infty}\frac{J_m(nx)}{n!}=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{x}{2}\right)^{m+2r}}{r!(m+r)!}n^{m+2r}=eJ_m(xS),\\ S^r=\sum_{k=1}^{r}S_2(r,k)\,.\end{array} \quad (79)$$

The above result can be extended to other summation formulae, as e.g. the formula, involving the Hermite polynomials below:

$$\sum_{n=0}^{\infty}\frac{H_m(nx,y)}{n!}=eH_m(xS,y). \quad (80)$$

The second example is relevant to the way of getting Rodriguez like expression for different from Hermite families of polynomials. We note indeed that being [1]

$$\exp(\lambda\hat{E}_x)f(x) = f(\exp(\lambda)x), \tag{81}$$

we can write the product of the exponents in the form of the following series:

$$\exp(-x)\exp(\lambda\hat{E}_x)\exp(x) = \exp((\exp(\lambda)-1)x) = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}b_n(x), \tag{82}$$

where $b_n(x)$– the Bell-type (binomial) polynomials. Now expand in series of λ the left hand side of the equation (82) and find after equating like power terms and using the identity [4] the following expressions for the binomial polynomials b_n:

$$\begin{gathered} b_n(x) = \exp(-x)\hat{E}_x^n \exp(x), \\ b_n(x) = e^{-x}\sum_{k=1}^{n} S_2(n,k)x^k \end{gathered} \quad . \tag{83}$$

Eventually, let's consider the example of the usage of the more general families of Stirling numbers to obtain summation formulae of the following type:

$$\eta_{m,p} = \sum_{n=0}^{\infty}\frac{n(n+p-1)(n+2p-2)...(n+p(m-1)-(m-1))}{n!}x^{n+pm-m}. \tag{84}$$

We note indeed that

$$\eta_{m,p} = (x^p D_x)^m e^x. \tag{85}$$

The use of the identity, employing the number triangle $S_2(p;m,k)$ [5]

$$\left(x^p\frac{d}{dx}\right)^m = \sum_{k=1}^{m} S_2(p;m,k)x^{k+(p-1)m}\hat{D}_x^k \tag{86}$$

eventually yields the following sum:

$$\eta_{m,p} = e^x\sum_{k=1}^{m} S_2(p;m,k)x^{k+(p-1)m} \quad , \tag{87}$$

the applications and interesting consequences of which will be explored elsewhere.

In conclusion we would like to underline that in the present paper we have developed the general approach, including the derivation of the Dobinsky-like formulae with the help of the method, combining features of combinatorial analysis and of the operational calculus. The examples considered in the present article and the results obtained demonstrate that it is in fact a powerful tool for investigation of a broad spectrum of physical problems related to the radiation by charged particles or quantum harmonic oscillators. In forthcoming publications we will discuss the application of this method to the solution of differential equations.

APPENDIX I

The evaluation of the following sum $S_{\mathrm{k}}(x,y)$:

$$S_k(x,y) = \exp\left(y\frac{\partial^2}{\partial x^2}\right)\left[x^k \exp(x)\right] \tag{88}$$

can be performed b by multiplying both sides by $\frac{t^k}{k!}$and then by summing up over k, which yields:

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}S_k(x,y) = \exp\left(y\frac{\partial^2}{\partial x^2}\right)\exp\left((t-1)x\right) \quad . \tag{89}$$

making use of the Crofton operational identity [1]

$$\exp\left(y\frac{\partial^m}{\partial x^m}\right)f(x) = f\left(x + my\frac{\partial^{m-1}}{\partial x^{m-1}}\right) \tag{90}$$

we obtain the following equation:

$$\begin{aligned}&\sum_{k=0}^{\infty}\tfrac{t^k}{k!}S_k(x,y) = \exp\left[(t+1)(x+2y\tfrac{\partial}{\partial x})\right] = \\ &= \exp\left[(t+1)x + y(t+1)^2\right] = \exp(x+y)\exp\left[(x+2y)t + yt^2\right]\end{aligned} \tag{91}$$

with the help of the Weyl decoupling formula. Now recalling the form of the Hermite polynomials generating function and applying it to the last expression in the eq. (91) we get the result (27). Moreover, we take note that

$$\eta_m(x,y) = \left(t\frac{d}{dt}\right)^m \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,y)|_{t=1} = \left(t\frac{d}{dt}\right)^m \exp(xt+yt^2)|_{t=1} \tag{92}$$

and make use of the eq. (4) together with the identity [11]

$$\left(\frac{d}{dt}\right)^m \exp(xt+yt^2) = H_m(x+2yt,y)\exp(xt+yt^2), \tag{93}$$

which allows the derivation of the eq. (28).

Along with the $H_n(x,y)$ the so called lacunary Hermite polynomials $H_n^{(m)}(x,y)$ are often exploited in different branches of Physics and of combinatorial analysis; they are defined as follows:

$$H_n^{(p)}(x,y) = n! \sum_{r=0}^{\left[\frac{n}{p}\right]} \frac{x^{n-pr} y^r}{(n-pr)!r!} \tag{94}$$

and their generating function writes in the following way:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(p)}(x,y) = \exp(xt+yt^p). \tag{95}$$

We can obtain the generalization of the eq. (28), namely

$$\eta_m(x,y|H^{(p)}) = \sum_{n=0}^{\infty} \frac{n^m}{n!} H_n^{(p)}(x,y), \tag{96}$$

using the same procedure as before and also using the following identity:

$$\begin{array}{l} \left(\frac{d}{dt}\right)^m \exp(P(x,t)) = H_m^{\{s\}_1^p}\left(\left\{\frac{1}{s!}\frac{\partial^s}{\partial t^s}P(x,t)\right\}_1^p\right)\exp(P(x,t)), \\ P(x,t) = xt+yt^p \end{array} . \tag{97}$$

Then the following result appears:

$$\eta_m^{(p)}(x,y) = \exp(x+y) \sum_{k=1}^{m} S_2(m,k) H_k^{\{s\}_1^p}\left(x+py, \left\{\left(\begin{array}{c} p \\ s \end{array}\right) y\right\}_2^p\right) \tag{98}$$

and it is also clear that from the operational point of view we get

$$\eta_m^{(p)}(x,y) = \exp\left(y\frac{\partial^p}{\partial x^p}\right)\eta_m\,(x|1)\,. \tag{99}$$

APPENDIX II

We will reconsider the Laguerre polynomials as element of the Appèl family by considering $y-$ *variable* and $x-$ parameter and thus writing

$$L_n(x,y) = C_0(x\hat{D}_y)y^n. \tag{100}$$

Accordingly we find:

$$\eta_m(x,y|L) = C_0(x\hat{D}_y)\sum_{k=1}^{m} S_2(m,k)y^k \exp(y). \tag{101}$$

The use of the integral representation

$$C_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp(\exp(i\phi) - x\exp(-i\phi))d\phi \tag{102}$$

results after some algebra in the eq. (34).

APPENDIX III

The truncated exponential polynomials belong to the Appèl family; they are specified by the following generating function:

$$\sum_{\substack{n=0 \\ |t|<1}}^{\infty} \frac{t^n}{n!}\bar{e}_n(x) = \frac{\exp(xt)}{1-t}, \bar{e}_n(x) = n!e_n(x) \tag{103}$$

and they can be written in terms of the associated Appèl operator as follows:

$$\bar{e}_n(x) = \frac{1}{1-\hat{D}_x}x^n. \tag{104}$$

The following summation formula ($|\alpha| < 1$) is true for them:

$$\eta_m(x|\bar{e}) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} n^m \bar{e}_n(x) = \frac{1}{1-\hat{D}_x}\sum_{k=1}^{m} S_2(m,k)(\alpha x)^k e^{\alpha x}. \tag{105}$$

The use of the Laplace transforms identity

$$\frac{1}{1- \hat{D}_x} = \int_0^\infty \exp(-s(1-\hat{D}_x))ds \tag{106}$$

finally yields

$$\eta_m(x|\bar{e}) = \exp(\alpha x)\sum_{k=1}^{m} \frac{S_2(m,k)}{(1-\alpha)^{k+1}}\alpha^k \bar{e}_k((1-\alpha)x). \tag{107}$$

APPENDIX IV

The derivation of the eq. (50) is easily achieved, once we have specified the action of the exponential operator

$$\hat{E} = \exp\left(|y|\frac{\partial^2}{\partial x^2} + |\tau|\frac{\partial^2}{\partial x \partial z} + |w|\frac{\partial^2}{\partial z^2}\right) \tag{108}$$

on $x^m z^n$. Therefore, we are looking for a generalized version of the Gauss-Weierstrass transform. To this end we recast our operator in the following form (we will omit in the following the absolute value sign, but it must be understood that $y > 0$, $\tau > 0$, $z > 0$):

$$\begin{array}{l} \hat{E} = \exp\left(y\left(\frac{\partial}{\partial x} + \frac{1}{2}\frac{\tau}{y}\frac{\partial}{\partial z}\right)^2 + \frac{\Delta}{y}\frac{\partial^2}{\partial z^2}\right), \\ \Delta = wy - \frac{\tau^2}{4} \end{array} . \tag{109}$$

With this restatement and using the following integral presentation:

$$\exp(\lambda \hat{c}^2) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp(-\sigma^2 + 2\sigma\sqrt{\lambda}\hat{c})d\sigma \tag{110}$$

we obtain:

$$\hat{E} = \frac{1}{\pi}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \exp(-(\xi^2 + \eta^2 - 2\sqrt{y}\xi\left(\frac{\partial}{\partial x} + \frac{1}{2}\frac{\tau}{y}\frac{\partial}{\partial z}\right) - 2\eta\sqrt{\frac{\Delta}{y}\frac{\partial}{\partial z}})). \tag{111}$$

The action of the above operator $\hat{E}$ on the function $F(x,y)$ is easily derived with the help of the well known identity:

$$\exp\left(a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}\right)F(x,y)=F(x+a,y+b), \tag{112}$$

which, along with the definition (19), yields the following result:

$$\hat{E}F(x,z)=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}d\xi\int\limits_{-\infty}^{\infty}d\eta\exp(-(\xi^2+\eta^2))F\left(x+2\sqrt{y}\xi,z+\frac{\tau}{\sqrt{y}}\xi+2\sqrt{\frac{\Delta}{y}}\eta\right). \tag{113}$$

Eventually, introducing new variables σ, ψ as follows:

$$\begin{array}{l} x+2\sqrt{y}\xi=\sigma, \\ z+\frac{\tau}{\sqrt{y}}\xi+2\sqrt{\frac{\Delta}{y}}\eta=\psi \end{array}, \tag{114}$$

we obtain the two-variable extension of the Gauss-Weierstrass transform, namely

$$\begin{array}{l} \hat{E}F(x,z)= \\ =\frac{1}{4\pi\sqrt{\Delta}}\int\limits_{-\infty}^{\infty}d\sigma\int\limits_{-\infty}^{\infty}d\psi\exp(-\frac{1}{4\Delta}(w(x-\sigma)^2-\tau(x-\sigma)(z-\psi)+y(z-\psi)^2))F(\sigma,\psi) \end{array}, \tag{115}$$

which can be easily exploited to get the result quoted in the paper.

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