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Comparative Growth Properties of Composite Meromorphic Functions

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Abstract

In the paper we study the comparative growth properties of composite meromorphic functions using $L^* - order$ and $L^* - type$ where $L = L(r)$ is a slowly changing function.

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1 Introduction, Definitions and Notations.

Let $L = L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [6] defined it in the following way:

Definition 1 ([6]) *A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon (> 0)$,*

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon} \text{ for } r \geq r(\varepsilon) \text{ and}$$

uniformly for $k\,(\geq 1)$.

If further, $L(r)$ *is differentiable, the above condition is equivalent to*

$$\lim_{r\to\infty}\frac{rL'(r)}{L(r)}=0\ .$$

Somasundaram and Thamizharasi [7] introduced the notions of $L-order$ and $L-type$ for entire functions defined in the open complex plane $\mathbb{C}$. The more generalised concept for $L-order$ and $L-type$ for entire and meromorphic functions are $L^*-order$ and L^*-type respectively. Their definitions are as follows:

Definition 2 ([7]) *The* $L^*-order$ $\rho_f^{L^*}$ *and the* $L^*-lower\ order$ $\lambda_f^{L^*}$ *of an entire function* f *are defined as*

$$\rho_f^{L^*}=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log\left[re^{L(r)}\right]}$$

and

$$\lambda_f^{L^*}=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log\left[re^{L(r)}\right]}\ ,$$

where $\log^{[k]}x=\log\left(\log^{[k-1]}x\right)$ *for* $k=1,2,3,\cdots$ *and* $\log^{[0]}x=x$.

When f *is meromorphic, one can easily verify that*

$$\rho_f^{L^*}=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}$$

and

$$\lambda_f^{L^*}=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}\ .$$

Definition 3 ([7]) *The* L^*-type $\sigma_f^{L^*}$ *of an entire function* f *is defined as follows:*

$$\sigma_f^{L^*}=\limsup_{r\to\infty}\frac{\log M(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}\ ,\ 0<\rho_f^{L^*}<\infty\ .$$

For meromorphic f,

$$\sigma_f^{L^*}=\limsup_{r\to\infty}\frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}\ ,\ 0<\rho_f^{L^*}<\infty\ .$$

Lakshminarasimhan [4] introduced the idea of the functions of L-bounded index. Later *Lahiri and Bhattacharjee* [5] worked on the entire functions of L-bounded index and of non uniform L-bounded index. In the paper we intend to establish some results on the comparative growth properties of composite entire and meromorphic functions using $L^*-order$ and L^*-type. We do not explain the standard definitions and notations in the theory of entire and meromorphic functions as those are available in [8] and [3].

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 **([1])** *If f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f) \ .$$

Lemma 2 **([2])** *Let f be meromorphic and g be entire and suppose that $0 < \mu \leq \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T(r, f \circ g) \geq T(\exp(r)^{\mu}, f) \ .$$

3 Theorems.

In this section we present the main results of the paper.

Theorem 1 *Let f be meromorphic and g be entire such that $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$. Then*

$$\limsup_{r \to \infty} \frac{\log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), f \circ g\right)}{\log T(\exp(r^{\mu}), f) + K(r, g; L)} = \infty \ ,$$

where $0 < \mu < \rho_g^{L^*}$ *and* $K(r, g; L) = \begin{cases} 0 \text{ if } r^{\mu} = o\left\{L\left(\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right)\right\} \\ \qquad\qquad\qquad\qquad \text{as } r \to \infty \\ L\left(\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right) \text{ otherwise .} \end{cases}$

Proof. Let $0 < \mu < \mu' < \rho_g^{L^*}$. Using the definition of $L^* -$ *lower order* we obtain in view of Lemma 2 for a sequence of values of R tending to infinity that

$$\begin{aligned}
\log T(R, f \circ g) &\geq \log T\left(\exp\left(R^{\mu'}\right), f\right) \\
\text{i.e., } \log T(R, f \circ g) &\geq \left(\lambda_f^{L^*} - \varepsilon\right) \log\left\{\exp\left(R^{\mu'}\right) . \exp L\left(\exp\left(R^{\mu'}\right)\right)\right\} \\
\text{i.e., } \log T(R, f \circ g) &\geq \left(\lambda_f^{L^*} - \varepsilon\right)\left\{R^{\mu'} + L\left(\exp\left(R^{\mu'}\right)\right)\right\} \\
\text{i.e., } \log T(R, f \circ g) &\geq \left(\lambda_f^{L^*} - \varepsilon\right)\left\{R^{\mu'}\left(1 + \frac{L(\exp(R^{\mu'}))}{R^{\mu'}}\right)\right\} \\
\text{i.e., } \log^{[2]} T(R, f \circ g) &\geq O(1) + \mu' \log R + \log\left\{1 + \frac{L(\exp(R^{\mu'}))}{R^{\mu'}}\right\} \ . \quad (1)
\end{aligned}$$

Now replacing R by $\exp\left(r^{\rho_g^{L^*}}\right)$ in (1) we get for a sequence of values of r tending to infinity that

$$\log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), f\circ g\right) \geq O(1) + \mu' \log\left(\exp\left(r^{\rho_g^{L^*}}\right)\right) + \log\left[1 + \frac{L\left\{\exp\left(\left(\exp\left(r^{\rho_g^{L^*}}\right)\right)^{\mu'}\right)\right\}}{\left\{\exp\left(r^{\rho_g^{L^*}}\right)\right\}^{\mu'}}\right]$$

$$i.e., \quad \log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), f\circ g\right) \geq O(1) + \mu' r^{\rho_g^{L^*}} + \log\left[1 + \frac{L\left\{\exp\left(\left(\exp\left(r^{\rho_g^{L^*}}\right)\right)^{\mu'}\right)\right\}}{\left\{\exp\left(r^{\rho_g^{L^*}}\right)\right\}^{\mu'}}\right]$$

$$i.e., \quad \log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), f\circ g\right) \geq O(1) + \mu' r^{\rho_g^{L^*}} + \log\left[1 + \frac{L\left\{\exp\left(\exp\left(\mu' r^{\rho_g^{L^*}}\right)\right)\right\}}{\exp\left(\mu' r^{\rho_g^{L^*}}\right)}\right]$$

$$i.e., \quad \log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), f\circ g\right) \geq O(1) + \mu' r^{\rho_g^{L^*}} + L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\} - \log\left[\exp\left\{L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\}\right\}\right] + \log\left[1 + \frac{L\left\{\exp\left(\exp\left(\mu' r^{\rho_g^{L^*}}\right)\right)\right\}}{\exp\left(\mu' r^{\rho_g^{L^*}}\right)}\right]$$

$$i.e., \quad \log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), f\circ g\right) \geq O(1) + \mu' r^{\rho_g^{L^*}} + L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\} + \log\left[\frac{1}{\exp\left\{L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\}\right\}} + \frac{L\left\{\exp\left(\exp\left(\mu' r^{\rho_g^{L^*}}\right)\right)\right\}}{\exp\left\{L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\}\right\} + \exp\left(\mu' r^{\rho_g^{L^*}}\right)}\right]$$

$$i.e., \ \log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), f\circ g\right) \geq O(1) + \mu' r^{\left(\rho_g^{L^*}-\mu\right)}.r^{\mu} + L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\} . \quad (2)$$

From the definition of $L^* - order$, we obtain for all sufficiently large values of R' that

$$\begin{aligned} \log T(R', f) &\leq \left(\rho_f^{L^*}+\varepsilon\right)\log\left\{R' e^{L(R')}\right\} \\ i.e., \ \log T(R', f) &\leq \left(\rho_f^{L^*}+\varepsilon\right)\left\{\log R' + L(R')\right\} . \quad (3) \end{aligned}$$

Putting $R' = \exp(r^{\mu})$ in (3) we obtain for all sufficiently large values of r that

$$\begin{aligned} \log T(\exp(r^{\mu}), f) &\leq \left(\rho_f^{L^*}+\varepsilon\right)\left\{\log(\exp(r^{\mu})) + L(\exp(r^{\mu}))\right\} \\ i.e., \ \log T(\exp(r^{\mu}), f) &\leq \left(\rho_f^{L^*}+\varepsilon\right)\left\{r^{\mu} + L(\exp(r^{\mu}))\right\} \end{aligned}$$

$$i.e., \ \frac{\log T(\exp(r^{\mu}), f) - \left(\rho_f^{L^*}+\varepsilon\right) L(\exp(r^{\mu}))}{\left(\rho_f^{L^*}+\varepsilon\right)} \leq r^{\mu} . \quad (4)$$

Now from (2) and (4) it follows for a sequence of values of r tending to infinity that

$$\begin{aligned} &\log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), f\circ g\right) \\ &\geq O(1) + \left(\frac{\mu' r^{\left(\rho_g^{L^*}-\mu\right)}}{\rho_f^{L^*}+\varepsilon}\right)\left[\log T(\exp(r^{\mu}), f) - \left(\rho_f^{L^*}+\varepsilon\right) L(\exp(r^{\mu}))\right] \\ &+ L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\} \quad (5) \end{aligned}$$

$$\begin{aligned} i.e., \ \frac{\log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), f\circ g\right)}{\log T(\exp(r^{\mu}), f)} &\geq \frac{L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\} + O(1)}{\log T(\exp(r^{\mu}), f)} \\ &+ \frac{\mu' r^{\left(\rho_g^{L^*}-\mu\right)}}{\rho_f^{L^*}+\varepsilon}\left\{1 - \frac{\left(\rho_f^{L^*}+\varepsilon\right) L(\exp(r^{\mu}))}{\log T(\exp(r^{\mu}), f)}\right\} . \quad (6) \end{aligned}$$

Again from (5) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), f \circ g\right)}{\log T\left(\exp\left(r^{\mu}\right), f\right) + L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\}} \geq \frac{O(1) - \mu' r^{\left(\rho_g^{L^*} - \mu\right)} L\left(\exp\left(r^{\mu}\right)\right)}{\log T\left(\exp\left(r^{\mu}\right), f\right) + L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\}} + \frac{\left(\frac{\mu' r^{\left(\rho_g^{L^*} - \mu\right)}}{\rho_f^{L^*} + \varepsilon}\right) \log T\left(\exp\left(r^{\mu}\right), f\right)}{\log T\left(\exp\left(r^{\mu}\right), f\right) + L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\}} + \frac{L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\}}{\log T\left(\exp\left(r^{\mu}\right), f\right) + L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\}}$$

i.e.,

$$\frac{\log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), f \circ g\right)}{\log T\left(\exp\left(r^{\mu}\right), f\right) + L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\}} \geq \frac{\frac{O(1) - \mu' r^{\left(\rho_g^{L^*} - \mu\right)} L\left(\exp\left(r^{\mu}\right)\right)}{L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\}}}{\frac{\log T\left(\exp\left(r^{\mu}\right), f\right)}{L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\}} + 1} + \frac{\left(\frac{\mu' r^{\left(\rho_g^{L^*} - \mu\right)}}{\rho_f^{L^*} + \varepsilon}\right) \log T\left(\exp\left(r^{\mu}\right), f\right)}{1 + \frac{L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\}}{\log T\left(\exp\left(r^{\mu}\right), f\right)}} + \frac{1}{1 + \frac{\log T\left(\exp\left(r^{\mu}\right), f\right)}{L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\}}} \,. \tag{7}$$

Case I. If $r^{\mu} = o\left\{L\left(\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right)\right\}$ then it follows from (6) that

$$\limsup_{r \to \infty} \frac{\log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), f \circ g\right)}{\log T\left(\exp\left(r^{\mu}\right), f\right)} = \infty \,.$$

Case II. $r^{\mu} \neq o\left\{L\left(\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right)\right\}$ then two sub cases may arise.

Sub case (a). If $L\left(\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right) = o\left\{\log T\left(\exp\left(r^{\mu}\right), f\right)\right\}$, then we get from (7) that

$$\limsup_{r \to \infty} \frac{\log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), f \circ g\right)}{\log T\left(\exp\left(r^{\mu}\right), f\right) + L\left(\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right)} = \infty \,.$$

Sub case (b). If $L\left(\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right) \sim \log T\left(\exp\left(r^{\mu}\right), f\right)$ then

$$\lim_{r\to\infty} \frac{L\left\{\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right\}}{\log T\left(\exp\left(r^{\mu}\right), f\right)} = 1$$

and we obtain from (7) that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), f\circ g\right)}{\log T\left(\exp\left(r^{\mu}\right), f\right) + L\left(\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right)} = \infty \ .$$

Combining Case I and Case II we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), f\circ g\right)}{\log T\left(\exp\left(r^{\mu}\right), f\right) + K\left(r, g; L\right)} = \infty \ ,$$

where $K\left(r,g;L\right) = \begin{cases} 0 \text{ if } r^{\mu} = o\left\{L\left(\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right)\right\} \\ \hfill \text{as } r\to\infty \\ L\left(\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right) \text{ otherwise }. \end{cases}$

This proves the theorem. ■

Theorem 2 *If f be meromorphic and g be entire such that (i) $0 < \rho_g^{L^*} < \infty$, (ii) $0 < \sigma_g^{L^*} < \infty$, (iii) $\rho_{f\circ g}^{L^*} = \rho_g^{L^*}$ and (iv) $\sigma_{f\circ g}^{L^*} < \infty$, then*

$$\liminf_{r\to\infty} \frac{T\left(r, f\circ g\right)}{T\left(r,g\right)} \le \frac{\sigma_{f\circ g}^{L^*}}{\sigma_g^{L^*}} \le \limsup_{r\to\infty} \frac{T\left(r, f\circ g\right)}{T\left(r,g\right)} \ .$$

Proof. From the definition of $L^* - type$ we obtain for all sufficiently large values of r that

$$T\left(r, f\circ g\right) \le \left(\sigma_{f\circ g}^{L^*} + \varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_{f\circ g}^{L^*}} \tag{8}$$

and

$$T\left(r, g\right) \le \left(\sigma_g^{L^*} + \varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_g^{L^*}} \ . \tag{9}$$

Also for a sequence of values of r tending to infinity that

$$T\left(r, f\circ g\right) \ge \left(\sigma_{f\circ g}^{L^*} - \varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_{f\circ g}^{L^*}} \tag{10}$$

and

$$T\left(r, g\right) \ge \left(\sigma_g^{L^*} - \varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_g^{L^*}} \ . \tag{11}$$

Now from (8) and (11) it follows for a sequence of values of r tending to infinity that

$$\frac{T(r, f \circ g)}{T(r, g)} \leq \frac{\left(\sigma_{f \circ g}^{L^*} + \varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_{f \circ g}^{L^*}}}{\left(\sigma_g^{L^*} - \varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_g^{L^*}}} . \tag{12}$$

In view of the condition (iii) we get from (12) that

$$\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} \leq \frac{\sigma_{f \circ g}^{L^*} + \varepsilon}{\sigma_g^{L^*} - \varepsilon} .$$

As $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} \leq \frac{\sigma_{f \circ g}^{L^*}}{\sigma_g^{L^*}} . \tag{13}$$

Again from (9) and (10) we get for a sequence of values of r tending to infinity that

$$\frac{T(r, f \circ g)}{T(r, g)} \geq \frac{\left(\sigma_{f \circ g}^{L^*} - \varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_{f \circ g}^{L^*}}}{\left(\sigma_g^{L^*} + \varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_g^{L^*}}} . \tag{14}$$

Since $\rho_{f \circ g}^{L^*} = \rho_g^{L^*}$, we obtain from (14) that

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} \geq \frac{\sigma_{f \circ g}^{L^*} - \varepsilon}{\sigma_g^{L^*} + \varepsilon} .$$

As $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} \geq \frac{\sigma_{f \circ g}^{L^*}}{\sigma_g^{L^*}} . \tag{15}$$

Thus the theorem follows from (13) and (15). ■

Theorem 3 *Let f be meromorphic and g be entire such that $0 < \lambda_{f \circ g}^{L^*} \leq \rho_{f \circ g}^{L^*} < \infty$ and $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$. If $L\left(r^A\right) = o\left\{\log T\left(r^A, g\right)\right\}$ as $r \to \infty$ then for any positive number A,*

$$\frac{\lambda_{f \circ g}^{L^*}}{A\rho_g^{L^*}} \leq \liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, g) + L(r^A)} \leq \frac{\lambda_{f \circ g}^{L^*}}{A\lambda_g^{L^*}}$$
$$\leq \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, g) + L(r^A)} \leq \frac{\rho_{f \circ g}^{L^*}}{A\lambda_g^{L^*}} .$$

Proof. From the definition of $L^* - order$ and $L^* - lower\ order$ we have for all sufficiently large values of r that

$$\log T\left(r, f\circ g\right) \geq \left(\lambda_{f\circ g}^{L^*} - \varepsilon\right)\log\left\{re^{L(r)}\right\}\ , \tag{16}$$

$$\log T\left(r^A, g\right) \geq \left(\lambda_{g}^{L^*} - \varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\}\ , \tag{17}$$

$$\log T\left(r, f\circ g\right) \leq \left(\rho_{f\circ g}^{L^*} + \varepsilon\right)\log\left\{re^{L(r)}\right\} \tag{18}$$

and

$$\log T\left(r^A, g\right) \leq \left(\rho_{g}^{L^*} + \varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\}\ . \tag{19}$$

Also for a sequence of values of r tending to infinity we get that

$$\log T\left(r, f\circ g\right) \leq \left(\lambda_{f\circ g}^{L^*} + \varepsilon\right)\log\left\{re^{L(r)}\right\}\ , \tag{20}$$

$$\log T\left(r^A, g\right) \leq \left(\lambda_{g}^{L^*} + \varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\}\ , \tag{21}$$

$$\log T\left(r, f\circ g\right) \geq \left(\rho_{f\circ g}^{L^*} - \varepsilon\right)\log\left\{re^{L(r)}\right\} \tag{22}$$

and

$$\log T\left(r^A, g\right) \geq \left(\rho_{g}^{L^*} - \varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\}\ . \tag{23}$$

From (16) we obtain for all sufficiently large values of r that

$$\begin{aligned} \log T\left(r, f\circ g\right) &\geq \left(\lambda_{f\circ g}^{L^*} - \varepsilon\right)\left\{\log r + L\left(r\right)\right\} \\ i.e.,\ \log T\left(r, f\circ g\right) &\geq \left(\lambda_{f\circ g}^{L^*} - \varepsilon\right)\left\{\log r + \frac{1}{A}L\left(r^A\right)\right\} \\ &\quad + \left(\lambda_{f\circ g}^{L^*} - \varepsilon\right)\left\{L\left(r\right) - \frac{1}{A}L\left(r^A\right)\right\}\ . \end{aligned} \tag{24}$$

Again from (19) we get for all sufficiently large values of r that

$$\begin{aligned} \log T\left(r^A, g\right) &\leq \left(\rho_{g}^{L^*} + \varepsilon\right)\left\{A\log r + L\left(r^A\right)\right\} \\ i.e.,\ \frac{\log T\left(r^A, g\right)}{A\left(\rho_{g}^{L^*} + \varepsilon\right)} &\leq \log r + \frac{1}{A}L\left(r^A\right)\ . \end{aligned} \tag{25}$$

Now from (24) and (25) it follows for all sufficiently large values of r that

$$\log T(r, f\circ g) \geq \frac{\left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)}{A\left(\rho_g^{L^*}+\varepsilon\right)}\log T\left(r^A, g\right) + \left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}$$

$$i.e., \ \frac{\log T(r, f\circ g)}{\log T(r^A, g)+L(r^A)} \geq \frac{\left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)}{A\left(\rho_g^{L^*}+\varepsilon\right)}\cdot\frac{\log T\left(r^A, g\right)}{\log T(r^A, g)+L(r^A)} + \frac{\left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}}{\log T(r^A, g)+L(r^A)}$$

$$i.e., \ \frac{\log T(r, f\circ g)}{\log T(r^A, g)+L(r^A)} \geq \frac{\frac{\left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)}{A\left(\rho_g^{L^*}+\varepsilon\right)}}{1+\frac{L(r^A)}{\log T(r^A, g)}} + \frac{\left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)\left\{\frac{L(r)}{L(r^A)}-\frac{1}{A}\right\}}{1+\frac{\log T(r^A, g)}{L(r^A)}} \ . \quad (26)$$

Since $L\left(r^A\right) = o\left\{\log T\left(r^A, g\right)\right\}$ as $r\to\infty$, it follows from (26) that

$$\liminf_{r\to\infty}\frac{\log T(r, f\circ g)}{\log T(r^A, g)+L(r^A)} \geq \frac{\left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)}{A\left(\rho_g^{L^*}+\varepsilon\right)} \ . \quad (27)$$

As $\varepsilon\,(>0)$ is arbitrary, we get from (27) that

$$\liminf_{r\to\infty}\frac{\log T(r, f\circ g)}{\log T(r^A, g)+L(r^A)} \geq \frac{\lambda_{f\circ g}^{L^*}}{A\rho_g^{L^*}} \ . \quad (28)$$

Again from (17) we obtain for all sufficiently large values of r that

$$\begin{aligned}\log T\left(r^A, g\right) &\geq \left(\lambda_g^{L^*}-\varepsilon\right)\left\{A\log r + L\left(r^A\right)\right\}\\ i.e., \ \frac{\log T\left(r^A, g\right)}{A\left(\lambda_g^{L^*}-\varepsilon\right)} &\geq \log r + \frac{1}{A}L\left(r^A\right) \ . \quad (29)\end{aligned}$$

From (20) we get for a sequence of values of r tending to infinity that

$$\begin{aligned}\log T(r, f\circ g) &\leq \left(\lambda_{f\circ g}^{L^*}+\varepsilon\right)\{\log r + L(r)\}\\ i.e., \ \log T(r, f\circ g) &\leq \left(\lambda_{f\circ g}^{L^*}+\varepsilon\right)\left\{\log r + \frac{1}{A}L\left(r^A\right)\right\}\\ &\quad + \left(\lambda_{f\circ g}^{L^*}+\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\} \ . \quad (30)\end{aligned}$$

From (29) and (30) it follows for a sequence of values of r tending to infinity that

$$\log T(r, f\circ g) \leq \frac{\left(\lambda_{f\circ g}^{L^*}+\varepsilon\right)}{A\left(\lambda_{g}^{L^*}-\varepsilon\right)}\log T\left(r^A, g\right) + \left(\lambda_{f\circ g}^{L^*}+\varepsilon\right)\left\{L(r) - \frac{1}{A}L\left(r^A\right)\right\}$$

$$i.e.,\ \frac{\log T(r, f\circ g)}{\log T(r^A, g) + L(r^A)} \leq \frac{\left(\lambda_{f\circ g}^{L^*}+\varepsilon\right)}{A\left(\lambda_{g}^{L^*}-\varepsilon\right)}\cdot\frac{\log T\left(r^A, g\right)}{\log T(r^A, g) + L(r^A)} + \frac{\left(\lambda_{f\circ g}^{L^*}+\varepsilon\right)\left\{L(r) - \frac{1}{A}L\left(r^A\right)\right\}}{\log T(r^A, g) + L(r^A)}$$

$$i.e.,\ \frac{\log T(r, f\circ g)}{\log T(r^A, g) + L(r^A)} \leq \frac{\frac{\left(\lambda_{f\circ g}^{L^*}+\varepsilon\right)}{A\left(\lambda_{g}^{L^*}-\varepsilon\right)}}{1+\frac{L(r^A)}{\log T(r^A, g)}} + \frac{\left(\lambda_{f\circ g}^{L^*}+\varepsilon\right)\left\{\frac{L(r)}{L(r^A)} - \frac{1}{A}\right\}}{1+\frac{\log T(r^A, g)}{L(r^A)}}\ . \qquad (31)$$

As $L\left(r^A\right) = o\left\{\log T\left(r^A, g\right)\right\}$ as $r\to\infty$ we get from (31) that

$$\liminf_{r\to\infty}\frac{\log T(r, f\circ g)}{\log T(r^A, g) + L(r^A)} \leq \frac{\left(\lambda_{f\circ g}^{L^*}+\varepsilon\right)}{A\left(\lambda_{g}^{L^*}-\varepsilon\right)}\ . \qquad (32)$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (32) that

$$\liminf_{r\to\infty}\frac{\log T(r, f\circ g)}{\log T(r^A, g) + L(r^A)} \leq \frac{\lambda_{f\circ g}^{L^*}}{A\lambda_{g}^{L^*}}\ . \qquad (33)$$

Also from (21) we obtain for a sequence of values of r tending to infinity that

$$\log T\left(r^A, g\right) \leq \left(\lambda_{g}^{L^*}+\varepsilon\right)\left\{A\log r + L\left(r^A\right)\right\}$$

$$i.e.,\ \frac{\log T\left(r^A, g\right)}{A\left(\lambda_{g}^{L^*}+\varepsilon\right)} \leq \log r + \frac{1}{A}L\left(r^A\right)\ . \qquad (34)$$

Now from (16) and (34) we get for a sequence of values of r tending to infinity that

$$\log T(r, f\circ g) \geq \left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)\{\log r + L(r)\}$$

$$i.e.,\ \log T(r, f\circ g) \geq \left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)\left\{\log r + \frac{1}{A}L\left(r^A\right)\right\} + \left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)\left\{L(r) - \frac{1}{A}L\left(r^A\right)\right\}$$

$$i.e.,\ \log T(r,f\circ g)\geq \frac{\left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)}{A\left(\lambda_g^{L^*}+\varepsilon\right)}\log T\left(r^A,g\right) + \left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}$$

$$i.e.,\ \frac{\log T(r,f\circ g)}{\log T(r^A,g)+L(r^A)}\geq \frac{\left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)}{A\left(\lambda_g^{L^*}+\varepsilon\right)}\cdot\frac{\log T\left(r^A,g\right)}{\log T(r^A,g)+L(r^A)} + \frac{\left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}}{\log T(r^A,g)+L(r^A)}$$

$$i.e.,\ \frac{\log T(r,f\circ g)}{\log T(r^A,g)+L(r^A)}\geq \frac{\frac{\left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)}{A\left(\lambda_g^{L^*}+\varepsilon\right)}}{1+\frac{L(r^A)}{\log T(r^A,g)}}+\frac{\left(\lambda_{f\circ g}^{L^*}-\varepsilon\right)\left\{\frac{L(r)}{L(r^A)}-\frac{1}{A}\right\}}{1+\frac{\log T(r^A,g)}{L(r^A)}}\ . \quad (35)$$

In view of the condition $L\left(r^A\right)=o\left\{\log T\left(r^A,g\right)\right\}$ as $r\to\infty$ we obtain from (35) that

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{\log T(r^A,g)+L(r^A)}\geq\frac{\lambda_{f\circ g}^{L^*}-\varepsilon}{A\left(\lambda_g^{L^*}+\varepsilon\right)}\ . \quad (36)$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (36) that

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{\log T(r^A,g)+L(r^A)}\geq\frac{\lambda_{f\circ g}^{L^*}}{A\lambda_g^{L^*}}\ . \quad (37)$$

From (18) we obtain for all large values of r that

$$\log T(r,f\circ g)\leq\left(\rho_{f\circ g}^{L^*}+\varepsilon\right)\{\log r+L(r)\}$$

$$i.e.,\ \log T(r,f\circ g)\leq\left(\rho_{f\circ g}^{L^*}+\varepsilon\right)\left\{\log r+\frac{1}{A}L\left(r^A\right)\right\} + \left(\rho_{f\circ g}^{L^*}+\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}\ . \quad (38)$$

Again from (17) we get for all large values of r that

$$\log T\left(r^A,g\right)\geq\left(\lambda_g^{L^*}-\varepsilon\right)\left\{A\log r+L\left(r^A\right)\right\}$$

$$i.e.,\ \frac{\log T\left(r^A,g\right)}{A\left(\lambda_g^{L^*}-\varepsilon\right)}\geq\log r+\frac{1}{A}L\left(r^A\right)\ . \quad (39)$$

Combining (38) and (39) it follows for all large values of r that

$$\log T(r, f\circ g) \le \frac{\left(\rho_{f\circ g}^{L^*}+\varepsilon\right)}{A\left(\lambda_g^{L^*}-\varepsilon\right)}\log T\left(r^A, g\right)+\left(\rho_{f\circ g}^{L^*}+\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}$$

$$i.e., \ \frac{\log T(r, f\circ g)}{\log T(r^A, g)+L(r^A)} \le \frac{\rho_{f\circ g}^{L^*}+\varepsilon}{A\left(\lambda_g^{L^*}-\varepsilon\right)}\cdot\frac{\log T\left(r^A, g\right)}{\log T(r^A, g)+L(r^A)} + \frac{\left(\rho_{f\circ g}^{L^*}+\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}}{\log T(r^A, g)+L(r^A)}$$

$$i.e., \ \frac{\log T(r, f\circ g)}{\log T(r^A, g)+L(r^A)} \le \frac{\frac{\rho_{f\circ g}^{L^*}+\varepsilon}{A\left(\lambda_g^{L^*}-\varepsilon\right)}}{1+\frac{L(r^A)}{\log T(r^A, g)}} + \frac{\left(\rho_{f\circ g}^{L^*}+\varepsilon\right)\left\{\frac{L(r)}{L(r^A)}-\frac{1}{A}\right\}}{1+\frac{\log T(r^A, g)}{L(r^A)}} \ . \quad (40)$$

Using $L\left(r^A\right)=o\left\{\log T\left(r^A, g\right)\right\}$ as $r\to\infty$ we obtain from (40) that

$$\limsup_{r\to\infty}\frac{\log T(r, f\circ g)}{\log T(r^A, g)+L(r^A)} \le \frac{\rho_{f\circ g}^{L^*}+\varepsilon}{A\left(\lambda_g^{L^*}-\varepsilon\right)} \ . \quad (41)$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from (41) that

$$\limsup_{r\to\infty}\frac{\log T(r, f\circ g)}{\log T(r^A, g)+L(r^A)} \le \frac{\rho_{f\circ g}^{L^*}}{A\lambda_g^{L^*}} \ . \quad (42)$$

Thus the theorem follows from (28), (33), (37) and (42). ■

Theorem 4 *Let f be meromorphic and g be entire such that $0<\rho_g^{L^*}<\rho_f^{L^*}<\infty$. Then*

$$\liminf_{r\to\infty}\frac{\log\{T(r, f\circ g)\log M(r, g)\}}{T(r, f)\cdot K(r, g; L)}=0 \ ,$$

where $K(r, g; L)=\begin{cases} 1 \ \textit{if } L(M(r, g))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \ \textit{as } r\to\infty \\ \qquad \textit{and for some } \alpha<\rho_f^{L^*} \\ L(M(r, g)) \ \textit{otherwise.}\end{cases}$

Proof. In view of Lemma 1 we have for all sufficiently large values of r that

$$\begin{aligned} T(r, f\circ g)\log M(r, g) &\le \{1+o(1)\}T(r, g)T(M(r, g), f) \\ i.e., \ \log\{T(r, f\circ g)\log M(r, g)\} &\le \log\{1+o(1)\}+\log T(r, g) \\ &\quad +\log T(M(r, g), f) \\ &= o(1)+\log T(r, g)+\log T(M(r, g), f). \end{aligned} \quad (43)$$

From the definition of $L^* - order$, we get for all sufficiently large values of r that

$$\begin{aligned} \log T(r,g) &\leq \left(\rho_g^{L^*} + \varepsilon\right) \log\left[re^{L(r)}\right] \\ i.e., \ \log T(r,g) &\leq \left(\rho_g^{L^*} + \varepsilon\right) \left[\log r + L(r)\right] \end{aligned} \tag{44}$$

and

$$\begin{aligned} \log^{[2]} M(r,g) &\leq \left(\rho_g^{L^*} + \varepsilon\right) \log\left[re^{L(r)}\right] \\ i.e., \ \log M(r,g) &\leq \left[re^{L(r)}\right]^{\left(\rho_g^{L^*} + \varepsilon\right)} . \end{aligned} \tag{45}$$

Also for a sequence of values of r tending to infinity it follows that

$$\begin{aligned} \log T(r,f) &\geq \left(\rho_f^{L^*} - \varepsilon\right) \log\left[re^{L(r)}\right] \\ i.e., \ T(r,f) &\geq \left[re^{L(r)}\right]^{\left(\rho_f^{L^*} - \varepsilon\right)} . \end{aligned} \tag{46}$$

Now from (43), (44) and (45) we obtain for all sufficiently large values of r that

$$\begin{aligned} \log\{T(r,f\circ g)\log M(r,g)\} \leq o(1) &+ \left(\rho_g^{L^*} + \varepsilon\right)\left[\log r + L(r)\right] \\ &+ \left(\rho_f^{L^*} + \varepsilon\right)\left[\log M(r,g) + L(M(r,g))\right] \end{aligned}$$

$$\begin{aligned} i.e., \ \log\{T(r,f\circ g)\log M(r,g)\} &\leq o(1) + \left(\rho_g^{L^*} + \varepsilon\right)\left[\log r + L(r)\right] \\ &+ \left(\rho_f^{L^*} + \varepsilon\right)\left[\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*} + \varepsilon\right)} + L(M(r,g))\right] . \end{aligned} \tag{47}$$

Now from (46) and (47) we get for a sequence of values of r tending to infinity that

$$\begin{aligned} \frac{\log\{T(r,f\circ g)\log M(r,g)\}}{T(r,f)} &\leq \frac{o(1) + \left(\rho_g^{L^*} + \varepsilon\right)\left[\log r + L(r)\right]}{T(r,f)} \\ &+ \frac{\left(\rho_f^{L^*} + \varepsilon\right)\left[\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*} + \varepsilon\right)} + L(M(r,g))\right]}{\left\{re^{L(r)}\right\}^{\left(\rho_f^{L^*} - \varepsilon\right)}} . \end{aligned} \tag{48}$$

Since $\rho_g^{L^*} < \rho_f^{L^*}$, we can choose $\varepsilon\,(> 0)$ in such a way that

$$\rho_g^{L^*} + \varepsilon < \rho_f^{L^*} - \varepsilon . \tag{49}$$

Case I. Let $L(M(r,g)) = o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some $\alpha < \rho_f^{L^*}$. As $\alpha < \rho_f^{L^*}$ we can choose $\varepsilon\,(>0)$ such that

$$\alpha < \rho_f^{L^*} - \varepsilon \ . \tag{50}$$

Since $L(M(r,g)) = o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ as $r \to \infty$ we get on using (50) that

$$\begin{aligned} \frac{L(M(r,g))}{r^{\alpha}e^{\alpha L(r)}} &\to 0 \ as\ r \to \infty \\ i.e.,\ \frac{L(M(r,g))}{\left[re^{L(r)}\right]^{\left(\rho_f^{L^*}-\varepsilon\right)}} &\to 0 \ as\ r \to \infty \ . \end{aligned} \tag{51}$$

Now in view of (48), (49) and (51) we get that

$$\liminf_{r\to\infty} \frac{\log\{T(r, f\circ g)\log M(r,g)\}}{T(r,f)} = 0 \ . \tag{52}$$

Case II. If $L(M(r,g)) \neq o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some $\alpha < \rho_f^{L^*}$ then we get from (48) that for a sequence of values of r tending to infinity,

$$\begin{aligned} \frac{\log\{T(r, f\circ g)\log M(r,g)\}}{T(r,f)\,L(M(r,g))} \le\ & \frac{o(1) + \left(\rho_g^{L^*}+\varepsilon\right)\left[\log\left\{re^{L(r)}\right\}\right]}{\left\{re^{L(r)}\right\}^{\left(\rho_f^{L^*}-\varepsilon\right)} L(M(r,g))} \\ & + \frac{\left(\rho_f^{L^*}+\varepsilon\right)\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)}}{\left\{re^{L(r)}\right\}^{\left(\rho_f^{L^*}-\varepsilon\right)} L(M(r,g))} \\ & + \frac{1}{\left\{re^{L(r)}\right\}^{\left(\rho_f^{L^*}-\varepsilon\right)} L(M(r,g))} \ . \end{aligned} \tag{53}$$

Now using (49) it follows from (53) that

$$\liminf_{r\to\infty} \frac{\log\{T(r, f\circ g)\log M(r,g)\}}{T(r,f)\,L(M(r,g))} = 0 \ . \tag{54}$$

Combining (52) and (54) we obtain that

$$\liminf_{r\to\infty} \frac{\log\{T(r, f\circ g)\log M(r,g)\}}{T(r,f)\cdot K(r,g;L)} = 0 \ ,$$

where $K(r,g;L) = \begin{cases} 1 \text{ if } L(M(r,g)) = o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \text{ as } r \to \infty \\ \qquad\qquad \text{and for some } \alpha < \rho_f^{L^*} \\ L(M(r,g)) \text{ otherwise.} \end{cases}$

Thus the theorem is established. ■

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