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The Schur Harmonic Convexity of Lehmer Means[1]

Weifeng Xia

School of Teacher Education
Huzhou Teachers College, Huzhou 313000, P.R. China

Yuming Chu

Department of Mathematics
Huzhou Teachers College, Huzhou 313000, P. R. China
chuyuming@hutc.zj.cn

Abstract

We prove that the Lehmer means $L_p(x,y)=(x^p+y^p)(x^{p-1}+y^{p-1})^{-1}$ are Schur harmonic convex (or concave, respectively) with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ if and only if $p\geq 0$ (or $p\leq 0$, respectively).

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1. Introduction

For $x,y>0$ and $p\in R$, the Lehmer mean values $L_p(x,y)$ were introduced by D.H.Lehmer[13] as follows:

$$L_p(x,y)=\frac{x^p+y^p}{x^{p-1}+y^{p-1}}. \tag{1.1}$$

It is easy to see that the Lehmer mean values $L_p(x,y)$ are continuous on the domain $\{(p;x,y): p\in R; x,y>0\}$ and differentiable with respect to $(x,y)\in(0,\infty)\times(0,\infty)$ for fixed $p\in R$. They are of symmetry between x

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and y. Many mean values are special cases of the Lehmer mean values, for example,

$$A(x,y) = \frac{x+y}{2} = L_1(x,y) \quad \text{is the arithmetic mean,}$$
$$G(x,y) = \sqrt{xy} = L_{\frac{1}{2}}(x,y) \quad \text{is the geometric mean,}$$
$$H(x,y) = \frac{2xy}{x+y} = L_0(x,y) \quad \text{is the harmonic mean,}$$
$$\widetilde{H}(x,y) = \frac{x^2+y^2}{x+y} = L_2(x,y) \quad \text{is the anti-harmonic mean.}$$

Investigation of the elementary properties and inequalities for $L_p(x,y)$ has attracted the attention of a considerable number of mathematicians (see[2,3,11,14,16,20,21,22]). In this paper, we shall research the Schur harmonic convexity of Lehmer means with respect to $(x,y) \in (0,\infty)\times(0,\infty)$ for fixed $p \in R$.

For convenience of readers, we recall the notations and definitions as follows.

For $x = (x_1, x_2), y = (y_1, y_2) \in (0,\infty)\times(0,\infty)$ and $\alpha \in R$, we denote by

$$x + y = (x_1 + y_1, x_2 + y_2),$$

$$xy = (x_1y_1, x_2y_2),$$
$$\alpha x = (\alpha x_1, \alpha x_2)$$

and
$$\frac{1}{x} = (\frac{1}{x_1}, \frac{1}{x_2}).$$

Definition 1.1. A set $E_1 \subseteq R^2$ is called a convex set if $\frac{x+y}{2} \in E_1$ whenever $x, y \in E_1$. A set $E_2 \subseteq (0,\infty)\times(0,\infty)$ is called a harmonic convex set if $\frac{2xy}{x+y} \in E_2$ whenever $x, y \in E_2$.

It is easy to see that $E \subseteq (0,\infty)\times(0,\infty)$ is a harmonic convex set if and only if $\frac{1}{E} = \{\frac{1}{x} : x \in E\}$ is a convex set.

Definition 1.2. Let $E \subseteq R^2$ be a convex set, $f : E \to R$ is said to be a convex function on E if $f(\frac{x+y}{2}) \le \frac{f(x)+f(y)}{2}$ for all $x, y \in E$. Moreover, f is called a concave function if $-f$ is a convex function.

Definition 1.3. Let $E \subseteq (0,\infty)\times(0,\infty)$ be a harmonic convex set, $f : E \to (0,\infty)$ is called a harmonic convex(or harmonic concave, respectively) function on E if

$$f(\frac{2xy}{x+y}) \le (\text{or} \ge, \text{respectively}) \frac{2f(x)f(y)}{f(x)+f(y)}$$

for all $x, y \in E$.

Definition 1.2 and 1.3 have the following consequences.

Remark 1.1. If $E_1 \subseteq (0, \infty) \times (0, \infty)$ is a harmonic convex set and $f : E_1 \to (0, \infty)$ is a harmonic convex function,then

$$F(x) = \frac{1}{f(\frac{1}{x})} : \frac{1}{E_1} \to (0, \infty)$$

is a concave function. Conversely, if $E_2 \subseteq (0, \infty) \times (0, \infty)$ is a convex set and $F : E_2 \to (0, \infty)$ is a convex function, then

$$f(x) = \frac{1}{F(\frac{1}{x})} : \frac{1}{E_2} \to (0, \infty)$$

is a harmonic concave function.

Definition 1.4. Let $E \subseteq R^2$ be a set. A function $F : E \to R$ is called a Schur convex function on E if

$$F(x_1, x_2) \leq F(y_1, y_2)$$

for each pair of two-tuples $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in E, such that $x \prec y$, i.e.

$$x_{[1]} \leq y_{[1]}$$

and

$$x_{[1]} + x_{[2]} = y_{[1]} + y_{[2]},$$

where $x_{[i]}$ denotes the ith largest component in x. A function F is called a Schur concave function if $-F$ is a Schur convex function.

Definition 1.5. Let $E \subseteq (0, \infty) \times (0, \infty)$ be a set. A function $F : E \to R$ is called a Schur harmonic convex(or Schur harmonic concave, respectively) function on E if

$$F(x_1, x_2) \leq (\text{or} \geq, \text{respectively}) F(y_1, y_2)$$

for each pair of $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in E, such that $\frac{1}{x} \prec \frac{1}{y}$.

Definition 1.4 and 1.5 have the following consequences.

Remark 1.2. Let $E \subseteq (0, \infty) \times (0, \infty)$ be a set, and $H = \frac{1}{E} = \{\frac{1}{x} : x \in E\}$, then $f : E \to (0, \infty)$ is a Schur harmonic convex(or concave, respectively) function on E if and only if $\frac{1}{f(\frac{1}{x})}$ is a Schur concave(or Schur convex, respectively)function on H.

The theory of Schur convex functions in the sense of arithmetic mean is one of the most important tool in finding inequalities, it can be used in isoperemetric problems for polytopes[23], linear regression[19], and graphs and matrices[6]. As the special case of the generalized means, the harmonic means play an important role in Fourier analysis[18], approximation to limiting values[8], extremum problems[7], combinatorial probability[1], Gamma and bata functions[9], inequalities theory[4] and many other mathematics branches. Recently, G.D.Anderson, M.K.Vamanamurthy and M.Vuorinen[5] discussed an attractive class of inequalities, which arise from the notation of harmonic convexity.

The following Gini means $S(a, b; x, y)$ are the generalization of the Lehmer means $L_p(x, y)$, which were first introduced by C.Gini[10].

$$S(p,q;x,y) = \begin{cases} (\frac{x^p+y^p}{x^q+y^q})^{\frac{1}{p-q}}, & p \neq q, \\ \exp(\frac{x^p \log x + y^p \log y}{x^p+y^p}), & p = q \neq 0, \\ \sqrt{xy}, & p = q = 0. \end{cases}$$

It is obvious that $L_p(x, y) = S(p, p-1; x, y)$. For the Schur convexity of the Gini means $S(p, q; x, y)$, J.Sándor [17] proved that the Gini mean values $S(p, q; x, y)$ are Schur convex on $(-\infty, 0] \times (-\infty, 0]$, and Schur concave on $[0, \infty)\times[0, \infty)$ with respect to (a, b) for fixed x, y with $x, y > 0$ and $x \neq y$. C.Gu and H.N.Shi[12] discussed the Schur convexity and Schur geometric convexity of the Lehmer means $L_p(x, y)$ with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ for fixed p.

Our aim in what follows is to discuss the Schur harmonic convexity of the Lehmer mean values $L_p(x, y)$ with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ for fixed $p \in R$. Our main result is the following Theorem 1.1.

Theorem 1.1. The Lehmer mean values $L_p(x, y)$ are Schur harmonic convex with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ if and only if $p \geq 0$ and Schur harmonic concave with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ if and only if $p \leq 0$.

2. Lemmas and Proof of Theorem 1.1

First, we introduce and establish two lemmas, which are used in the proof of Theorem 1.1.

Lemma 2.1(see[15]) Let $E \subseteq R^2$ be a symmetric convex set with nonempty interior int$E$ and $\varphi : E \to R$ be a continuous symmetric function on E. If φ is differentiable on int$E$, then φ is Schur convex(or Schur concave, respectively) on E if and only if

$$(y-x)(\frac{\partial \varphi}{\partial y} - \frac{\partial \varphi}{\partial x}) \geq 0 (or \leq 0, respectively)$$

for all $(x, y) \in intE$.

Lemma 2.2. Let $E \subseteq (0, \infty) \times (0, \infty)$ be a symmetric harmonic convex set with nonempty interior int$E$ and $\varphi : E \to (0, \infty)$ be a continuous symmetric function on E. If φ is differentiable on int$E$, then φ is Schur harmonic convex(or concave, respectively) on E if and only if

$$(y-x)(y^2\frac{\partial\varphi}{\partial y} - x^2\frac{\partial\varphi}{\partial x}) \geq 0(or \leq 0, respectively)$$

for all $(x, y) \in \mathrm{int}E$.

Proof. Lemma 2.2 follows from Lemma 2.1 and Remark 1.2 together with the elementary computation.

Proof of Theorem 1.1. We use Lemma 2.2 to discuss the nonnegativity and nonpositivity of $(y-x)(y^2\frac{\partial L_p(x,y)}{\partial y} - x^2\frac{\partial L_p(x,y)}{\partial x})$ for all $(x, y) \in (0, \infty) \times (0, \infty)$ and for fixed $p \in R$. Since $L_p(x, y)$ is symmetric with respect to x and y. Without loss of generality we assume $y > x$ in the following discussion.

By simple computation, (1.1) leads to the following identity

$$\begin{aligned} & (y-x)(y^2\frac{\partial L_p(x,y)}{\partial y} - x^2\frac{\partial L_p(x,y)}{\partial x}) \\ = \ & \frac{y-x}{(x^{p-1}+y^{p-1})^2}[(y^{2p}-x^{2p}) + px^{p-1}y^{p-1}(y^2-x^2)]. \end{aligned} \tag{2.1}$$

If $p \geq 0$, then $(y-x)(y^2\frac{\partial L_p(x,y)}{\partial y} - x^2\frac{\partial L_p(x,y)}{\partial x}) \geq 0$ follows from (2.1) and $(x, y) \in (0, \infty) \times (0, \infty)$ together with the assumption $y > x$, this and Lemma 2.2 imply that $L_p(x, y)$ is Schur harmonic convex with respect to $(x, y) \in (0, \infty) \times (0, \infty)$.

If $p \leq 0$, then $(y-x)(y^2\frac{\partial L_p(x,y)}{\partial y} - x^2\frac{\partial L_p(x,y)}{\partial x}) \leq 0$ again follows from (2.1) and $(x, y) \in (0, \infty) \times (0, \infty)$ together with the assumption $y > x$, and we know that $L_p(x, y)$ is Schur harmonic concave with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ by Lemma 2.2.

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