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On the Numerical Analysis of the Ergodic Control Quasi-Variational Inequalities

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Abstract. This paper deals with the convergence of the standard finite element approximation of elliptic quasi-variational inequalities (QVI) when the discount factor (the zero order term) goes to zero. The proof combines a result due to P. L. Lions and B. Perthame [8] with standard L^∞- error estimate for elliptic QVIs.

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1. Introduction

It is well known that impulse control problems for reflected diffusion process may be solved by considering the solution of quasi-variational inequalities (QVI) with Neumann boundary conditions (see A. Bensoussan [1] and A. Bensoussan and J.L. Lions [2] for more details). A typical example is the following:

$$(1.1)\qquad \begin{cases} a(u_\alpha, v-v_\alpha)+\alpha(u_\alpha, v-v_\alpha)\geq (f\ , v-u_\alpha)\ \forall v\in H^1(\Omega) \\ v\leq Mu_\alpha;\ u_\alpha\in H^1(\Omega),\ u_\alpha\leq Mu_\alpha \end{cases}$$

where Ω is a given bounded smooth open set in $\mathbb{R}^n$, $\alpha>0$, f is a given function, M is an operator defined (for example) on $C(\bar{\Omega})$ by

$$(1.2)\qquad M\varphi(x)=k+\inf\varphi\left(x+\xi\right):\xi\geq 0,\ \ x+\xi\in\bar{\Omega},\ \text{ where } k>0$$

and assumed to map $C(\bar{\Omega})$ onto itself, $a(.,.)=\int_\Omega \nabla u.\nabla v.dx$, and $(.,.)$ denotes the L^2-inner product on Ω.

It has been proved that the long run average cost for this problem solves the ergodic QVI. More precisely, denoting by

$$\langle\omega\rangle=\frac{1}{\Omega}\int_\Omega \omega dx,\ \omega_\alpha=\ u_\alpha-\langle u_\alpha\rangle\ ,\ \text{and } \lambda_\alpha=\alpha\left\langle u_\alpha\right\rangle$$

P.L. Lions and B. Perthame ([8]) proved that the solution $(\omega_\alpha, \lambda_\alpha)$ of the QVI

$$(1.3)\quad \begin{cases} a(\omega_\alpha, v-\omega_\alpha) + \alpha(\omega_\alpha, v-\omega_\alpha) \geq (f-\lambda_\alpha\ , v-\omega_\alpha)\ \forall v \in H^1(\Omega) \\ v \leq M\omega_\alpha; \omega_\alpha \in H^1(\Omega), \omega_\alpha \leq M\omega_\alpha, \langle\omega_\alpha\rangle = 0 \end{cases}$$

converges to the solution of the ergodic control QVI

$$(1.4)\quad \begin{cases} a(\omega_0, v-\omega_0) \geq (f-\lambda_0, v-\omega_0)\, \forall v \in H^1(\Omega) \\ v \leq M\omega_0 \leq M\omega_0;\ \omega_0 \in H^1(\Omega),\ \omega_0 \leq M\omega_0,\ \ \langle\omega_0\rangle = 0 \end{cases}$$

as stated in the following theorem.

Theorem 1. *As α goes to 0^+, λ_α converges uniformly in $C(\bar{\Omega})$ to some constant λ_0, and ω_α converges uniformly in $C(\bar{\Omega})$ and strongly in $H^1(\Omega)$ to ω_0.Moreover, (λ_0, ω_0) is the unique solution of the quasi-variational inequality of ergodic control problem (1.4).*

In this paper, we establish the following convergence result:

$$(1.5)\quad \lim_{h\to 0}\ \|\omega_{\alpha h} - \omega_0\|_\infty = 0$$

and

$$(1.6)\quad \lim_{h\to 0}\ |\lambda_{\alpha h} - \lambda_0| = 0$$

where $\omega_{\alpha h}$ and $\lambda_{\alpha h}$ denote the piecewise linear approximations of ω_α and λ_α, respectively.

The proof combines theorem 1 with standard L^∞- error estimate for elliptic QVIs. (see [6], [7], [3], [4]).

More precisely, we first explicit the dependency of the L^∞- error estimate for QVI (1.1) on the parameter α, and then by a suitable choice of this parameter in terms of the mesh-size h, we derive the above convergence results.

It is worth mentioning that to the best of our knowledge, the present paper contains the first contribution to the numerical analysis of quasi-variational inequalities with vanishing zero order term.

2. Preliminaries

In this section, we shall characterize both the continuous and discrete solutions of (1.1) as fixed points of contractions.

2.1. **A contraction associated with QVI (1.1)**

. Let α be **fixed** in the open interval $]0\,,1[$ and set $\gamma = 1-\alpha < 1$. Then, one can easily see that problem (1.1) is equivalent to the following QVI:

$$(2.1)\quad \begin{cases} b(u_\alpha, v-v_\alpha) \geq (f+\gamma u_\alpha\ , v-v_\alpha)\ \forall v \in H^1(\Omega) \\ v \leq Mu_\alpha;\ u_\alpha \in Mu_\alpha, u_\alpha \leq Mu_\alpha \end{cases}$$

where

$$(2.2)\quad b(u,v) = a(u,v) + (u,v)$$

Thanks to [2], (1.1) or (2.1) has a unique solution. Also, notice that, as the bilinear form (2.2) is independent of α, the left hand-side of (2.1) is independent of α too.

Next we shall characterize u_α as the fixed point of a contraction. Indeed, consider the mapping

$$\begin{aligned} T : L^\infty(\Omega) &\longrightarrow L^\infty(\Omega) \\ w &\longrightarrow Tw = \zeta \end{aligned} \tag{2.3}$$

where ζ solves the QVI

$$\begin{cases} b(\zeta, v-\zeta) \geqq (f+\gamma w, v-\zeta) \forall v \in H^1(\Omega) \\ v \leq M\zeta;\ \zeta \in H^1(\Omega), \zeta \leq M\zeta \end{cases} \tag{2.4}$$

Let us denote this solution by $\zeta = \sigma(f+\gamma w,\ M\zeta)$.

Proposition 1. *Let α be **fixed** in the open interval $]0\,,1[$. Then, the mapping T is a contraction whose unique fixed point coincides with the solution of QVI (1.1).*

Proof. For w, $\widetilde{w}$ in $L^\infty(\Omega)$ we consider $\zeta = Tw$ and $\widetilde{\zeta} = T\widetilde{w}$ solutions to QVI (2.4) with right hand sides $F = f + \gamma w$ and $f + \gamma\widetilde{w}$, respectively. Let $\Phi = \| F - \widetilde{F} \|_\infty$. Then, since,

$$F \leq \widetilde{F} + \| F - \widetilde{F} \|_\infty$$

, making use of standard comparison results in elliptic QVIs, we have

$$\sigma(F,\ M\zeta) \leq \sigma(\widetilde{F} + (a_0(x)+\gamma)\Phi,\ M(\widetilde{\zeta}+\Phi)) = \sigma(\widetilde{F},\ M\widetilde{\zeta}) + \Phi$$

Thus

$$\zeta \leq \widetilde{\zeta} + \Phi$$

Also, interchanging the roles w and $\widetilde{w}$, we similarly get

$$\widetilde{\zeta} \leq \zeta + \Phi$$

Therefore

$$\| Tw - T\tilde{w} \|_\infty \leq \ \| F - \widetilde{F} \|_\infty \leq \gamma \| w - \widetilde{w} \|_\infty < \ \| w - \widetilde{w} \|_\infty$$

which completes the proof. □

Let Ω be decomposed into triangles and τ_h be the set of those elements; $h > 0$ is the mesh-size.We assume that Ω is polygonal and that the triangulation τ_h is regular and quasi-uniform. Let $\mathbb{V}_h$ denote the finite element space consisting of piecewise linear functions:

$$\mathbb{V}_h = \{v \in C(\bar{\Omega}) \cap H^1(\Omega) \text{ such that } v/_K \in P_1\}$$

where K is a triangle of τ_h, P_1 is the space of polynomials with degree ≤ 1, and $\{\varphi_i\}$; $i = 1, ..., m(h)$ be the basis functions of $\mathbb{V}_h$.

Let also r_h be the restriction operator defined by:

$$\forall v \in C(\bar{\Omega}) \cap H^1(\Omega),\ r_h v = \sum_{i=1}^{m(h} v_i \varphi_i$$

In the sequel we shall make use of **the discrete maximum principle** assumption (**d.m.p**), that is, we assume that the matrix $\mathbb{B}_{ij} = b(\varphi_i, \varphi_j)$, $1 \leq i, j \leq m(h)$ is an Matrix [5]. Consider the discrete QVI

$$\left\{ \begin{array}{l} a(u_{\alpha h}, v - u_{\alpha h}) + \alpha(u_{\alpha h}, v - u_{\alpha h}) \geq (f, v - u_{\alpha h})\ \forall v \in \mathbb{V}_h \\ v \leq r_h M u_{\alpha h};\ u_{\alpha h} \in \mathbb{V}_h,\ u_{\alpha h} \leq r_h M u_{\alpha h} \end{array} \right. \tag{2.5}$$

Thanks to ([6]), QVI (2.5) has a unique solution.

As in the continuous case, we shall characterize the solution of (2.5) as the unique fixed point of a contraction. Indeed, consider the discrete mapping.

2.2. **A contraction associated with QVI (2.5)**

. First, it is easy to see that $u_{\alpha h}$, the solution of (2.5), is also solution to the following QVI:

$$\left\{ \begin{array}{l} b(u_{\alpha h}, v - u_{\alpha h}) \geq (f + \gamma u_{\alpha h}, v - u_{\alpha})\ \forall v \in \mathbb{V}_h \\ v \leq r_h M u_{\alpha h};\ u_{\alpha h} \in \mathbb{V}_h,\ u_{\alpha h} \leq r_h M u_{\alpha h} \end{array} \right. \tag{2.6}$$

Now, let α be **fixed** in $]0\,,1[$ and consider the discrete mapping

$$\begin{aligned} T_h :& L^\infty_+(\Omega) \longrightarrow \mathbb{V}_h \\ & w \longrightarrow T_h w = \zeta_h \end{aligned} \tag{2.7}$$

where $\zeta_h = \sigma_h(f + \gamma w,\ M\zeta_h)$ is the unique solution of the following discrete QVI:

$$\left\{ \begin{array}{l} b(\zeta_h, v - \zeta_h) \geqq (f + \gamma w, v - \zeta_h)\ \forall v \in \mathbb{V}_h \\ v \leq r_h M\zeta_h;\ \zeta_h \in \mathbb{V}_h,\ \zeta_h \leq r_h M\zeta_h \end{array} \right. \tag{2.8}$$

Proposition 2. *Let α be **fixed** in the open interval $]0\,,1[$. Then, under the **dmp**, the mapping T_h is a contraction whose unique fixed point coincides with $u_{\alpha h}$, the solution of discrete QVI (2.5).*

Proof. It is exactly the same as that of proposition 1. □

3. Convergence of the approximation

3.1. **L^∞- error estimate for QVI (1.1)**

. Below we shall explicit the dependency of the L^∞-error estimate for QVI (1.1) on the parameter α. To this end, we begin by recalling standard results on L^∞- error estimates for elliptic QVIs. Let $g \in L^\infty(\Omega)$, and consider the elliptic QVI

$$\left\{ \begin{array}{l} b(\zeta, v - \zeta) \geq (g, v - \zeta)\ \forall v \in H^1(\Omega) \\ v \leq M\zeta;\ \zeta \in H^1(\Omega),\ \zeta \leq M\zeta \end{array} \right. \tag{3.1}$$

Problem (3.1) has a unique solution. Moreover $\zeta \in W^{2,p}(\Omega)$, $2 \leq p < \infty$.(see [2])

Theorem 2. *(cf.[7]) Let ζ_h denote its discrete counterpart of ζ.Then, there exists a constant C independent of h such that*

$$\|\zeta - \zeta_h\|_\infty \leq Ch^2 |\ln h|^3 \|g\|_\infty \tag{3.2}$$

Theorem 3. *Let u_α and $u_{\alpha h}$ be the solutions of (1.1) and (2.5), respectively. Then, there exists a constant C independent of both α and h such that*

$$\|u_\alpha - u_{\alpha h}\|_\infty \leq C.\alpha^{-2} h^2 |\log h|^3$$

Proof. From propositions 1 and 2, it is clear that

$$u_\alpha = Tu_\alpha = \sigma(f + \lambda u_\alpha, Mu_\alpha) \text{ and } u_{\alpha h} = T_h u_{\alpha h} = \sigma_h(f + \lambda u_{\alpha h}, Mu_{\alpha h})$$

Let also $U_{\alpha h} = T_h u_\alpha = \sigma_h(f + \lambda u_{\alpha h},\, MU_{\alpha h})$ be the solution of the QVI

$$\begin{cases} b(U_{\alpha h}, v - U_{\alpha h}) \geq (f + \lambda u_{\alpha h}, v - U_{\alpha h}) \ \forall v \in \mathbb{V}_h \\ v \leq U_{\alpha h}; U_{\alpha h} \in \mathbb{V}_h,\ U_{\alpha h} \leq r_h MU_{\alpha h} \end{cases}$$

Then, clearly, $U_{\alpha h}$ is nothing but the discrete counterpart of u_α. So, since the bilinear form (2.2) is independent of α, making use of (3.2), we have

$$\|T_h u_\alpha - Tu_\alpha\|_\infty \ \leq\ \|U_{\alpha h} - u_\alpha\|_\infty\ \leq Ch^2 |\log h|^3 \|f + \lambda u_\alpha\|_\infty$$

Hence, since $\|u_\alpha\|_\infty \leq \alpha^{-1}. \|f\|_\infty$, it follows that

$$\begin{aligned} \|u_\alpha - u_{\alpha h}\|_\infty &\leq\ \|u_\alpha - T_h u_\alpha\|_\infty + \|T_h u_\alpha - u_{\alpha h}\|_\infty \\ &\leq \|Tu_\alpha - T_h u_\alpha\|_\infty\ +\ \|T_h u_\alpha - T_h u_{\alpha h}\|_\infty \\ &\leq Ch^2 |\log h|^3 \|u_\alpha\|_\infty + \gamma \|u_\alpha - u_{\alpha h}\|_\infty \\ &\leq C\alpha^{-1} h^2 |\log h|^3 \|f\|_\infty + \gamma \|u_\alpha - u_{\alpha h}\|_\infty \end{aligned}$$

which yields the desired error estimate. □

3.2. **The Main Result.**

Theorem 4. *Under conditions of Theorems 1 and 3, we have*

$$i)\ \lim_{h\to 0} \|\omega_{\alpha h} - \omega_0\|_\infty = 0$$
$$ii)\ \lim_{h\to 0} |\lambda_{\alpha h} - \lambda_0| = 0$$

Proof. Since $\omega_\alpha = u_\alpha - \langle u_\alpha \rangle$ and $\omega_{\alpha h} = u_{\alpha h} - \langle u_{\alpha h} \rangle$,we have

$$\begin{aligned}
\|\omega_{\alpha h} - \omega_0\|_\infty &\leq \|\omega_{\alpha h} - \omega_\alpha\|_\infty + \|\omega_\alpha - \omega_0\|_\infty \\
&\leq \|u_\alpha - \langle u_\alpha \rangle - (u_{\alpha h} - \langle u_{\alpha h} \rangle)\|_\infty + \|\omega_\alpha - \omega_0\|_\infty \\
&\leq \|u_\alpha - u_{\alpha h}\|_\infty + (meas(\Omega))^{-1} \|u_\alpha - u_{\alpha h}\|_\infty + \|\omega_\alpha - \omega_0\|_\infty \\
&\leq Ch^2\alpha^{-2} |\log h|^3 + \|\omega_\alpha - \omega_0\|_\infty
\end{aligned}$$

So by taking $\alpha = h^{1/2}$, and then passing to the limit as h goes to zero, we get (i)

Now let us prove (ii). Indeed, since $\lambda_\alpha = \alpha \langle u_\alpha \rangle$ and $\lambda_{\alpha h} = \alpha \langle u_{\alpha h} \rangle$, we have

$$\begin{aligned}
|\lambda_{\alpha h} - \lambda_0| &\leq |\lambda_{\alpha h} - \lambda_\alpha| + |\lambda_\alpha - \lambda_0| \\
&\leq |\alpha \langle u_{\alpha h} \rangle - \alpha \langle u_\alpha \rangle| + |\lambda_\alpha - \lambda_0| \\
&\leq \alpha \,|\langle u_\alpha - u_{\alpha h} \rangle| + |\lambda_\alpha - \lambda_0| \\
&\leq Ch^2\alpha.\alpha^{-2} |\log h|^3 + |\lambda_\alpha - \lambda_0| \\
&\leq Ch^2.\alpha^{-1} |\log h|^3 + |\lambda_\alpha - \lambda_0|
\end{aligned}$$

Thus, by taking $\alpha = h^{1/2}$, and passing to the limit as h goes to zero, we obtain (ii). □

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