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On Closed Maps in BiČech Closure Spaces

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Abstract

The purpose of this paper is to study and investigate some properties of closed maps in biČech closure spaces.

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1 INTRODUCTION

Čech closure spaces were introduced by E. Čech [1] and then studied by many authors, see e.g. [3, 4, 6, 7]. In Čech's approach the operator satisfies idempotent condition among Kuratowski axioms. This condition need not hold for every subset of X. When this condition is also true, the operator becomes topological closure operator. Thus the concept of closure space is the generalization of a topological space. BiČech closure spaces were introduced by K. Chandrasekhara Rao , R. Gowri and V. Swaminathan [2]. In this paper we study and investigate some properties of closed maps in biČech closure spaces.

2 PRELIMINARIES

An operator $u : P(X) \rightarrow P(X)$ defined on the power set $P(X)$ of a set X satisfying the axioms :

(C1) $u\emptyset = \emptyset$,

(C2) $A \subseteq uA$ for every $A \subseteq X$, (C3) $u(A \cup B) = uA \cup uB$ for all $A, B \subseteq X$.

is called a *Čech closure operator* and the pair (X, u) is a *Čech closure space*. For short, the space will be noted by X as well, and called a *closure space*. A closure operator u on a set X is called *idempotent* if $uA = uuA$ for all $A \subseteq X$.

A subset A is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if its complement is closed. The empty set and the whole space are both open and closed.

A closure space (Y, v) is said to be a *subspace* of (X, u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$. If Y is closed in (X, u), then the subspace (Y, v) of (X, u) is said to be closed too.

Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

One can see that a map $f : (X, u) \to (Y, v)$ is continuous if and only if $uf^{-1}(B) \subseteq f^{-1}(vB)$ for every subset $B \subseteq Y$.

Clearly, if $f : (X, u) \to (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of (X, u) for every closed subset F of (Y, v).

Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *closed* (resp. *open*) if $f(F)$ is a closed (resp. open) subset of (Y, v) whenever F is a closed (resp. open) subset of (X, u).

The *product* of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$, is the closure space $\big(\prod\limits_{\alpha \in I} X_\alpha, u\big)$ where $\prod\limits_{\alpha \in I} X_\alpha$ denotes the cartesian product of sets X_α, $\alpha \in I$, and u is the closure operator generated by the projections $\pi_\alpha : \prod\limits_{\alpha \in I} X_\alpha \to X_\alpha$, $\alpha \in I$, i.e., is defined by $uA = \prod\limits_{\alpha \in I} u_\alpha \pi_\alpha(A)$ for each $A \subseteq \prod\limits_{\alpha \in I} X_\alpha$.

Clearly, if $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ is a family of closure spaces, then the projection map $\pi_\beta : \prod\limits_{\alpha \in I}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is closed and continuous for every $\beta \in I$.

Definition 2.1. *Two maps u_1 and u_2 from power set to itself are called* biČech closure operator *(simply biclosure operator) for if they satisfies the following properties:*

(i) $u_1\emptyset = \emptyset$ *and* $u_2\emptyset = \emptyset$,

(ii) $A \subseteq u_1A$ *and* $A \subseteq u_2A$ *for all* $A \subseteq X$,

(iii) $u_1(A \cup B) = u_1A \cup u_1B$ *and* $u_2(A \cup B) = u_2A \cup u_2B$ *for all* $A, B \subseteq X$.

A structure (X, u_1, u_2) *is called* biČech closure space.

Definition 2.2. *A subset A of a biČech closure space (X, u_1, u_2) is called closed if $u_1u_2A = A$. The complement of closed set is called open.*

Clearly, A is a closed subset of a biČech closure space (X,u_1,u_2) if and only if A is both a closed subset of (X,u_1) and (X,u_2).

Let A be a closed subset of a biČech closure space (X,u_1,u_2) . The following conditions are equivalent

(i) $u_2u_1A=A$,

(ii) $u_1A=A$, $u_2A=A$.

Proposition 2.3. *Let $\{(X_\alpha,u_\alpha^1,u_\alpha^2):\alpha\in I\}$ be a family of biČech closure spaces. Then F is a closed subset of $\prod_{\alpha\in I}(X_\alpha,u_\alpha^1,u_\alpha^2)$ if and only if F is both a closed subset of $\prod_{\alpha\in I}(X_\alpha,u_\alpha^1)$ and $\prod_{\alpha\in I}(X_\alpha,u_\alpha^2)$.*

Proof. Let F be a closed subset of $\prod_{\alpha\in I}(X_\alpha,u_\alpha^1,u_\alpha^2)$. Then $F=\prod_{\alpha\in I}u_\alpha^1\pi_\alpha\Big(\prod_{\alpha\in I}u_\alpha^2\pi_\alpha(F)\Big)$. Since $F\subseteq\prod_{\alpha\in I}u_\alpha^2\pi_\alpha(F)$, $\prod_{\alpha\in I}u_\alpha^1\pi_\alpha(F)\subseteq\prod_{\alpha\in I}u_\alpha^1\pi_\alpha\Big(\prod_{\alpha\in I}u_\alpha^2\pi_\alpha(F)\Big)=F$. Therefore, F is a closed subset of $\prod_{\alpha\in I}(X_\alpha,u_\alpha^1)$. Since $\prod_{\alpha\in I}u_\alpha^2\pi_\alpha(F)\subseteq\prod_{\alpha\in I}u_\alpha^2\pi_\alpha(F)$, $\prod_{\alpha\in I}u_\alpha^2\pi_\alpha(F)\subseteq\prod_{\alpha\in I}u_\alpha^1\pi_\alpha\Big(\prod_{\alpha\in I}u_\alpha^2\pi_\alpha(F)\Big)=F$. Consequently, F is a closed subset of $\prod_{\alpha\in I}(X_\alpha,u_\alpha^2)$.

Conversely, let F be a closed subset of $\prod_{\alpha\in I}(X_\alpha,u_\alpha^1)$ and $\prod_{\alpha\in I}(X_\alpha,u_\alpha^2)$. Then $F=\prod_{\alpha\in I}u_\alpha^1\pi_\alpha(F)$ and $F=\prod_{\alpha\in I}u_\alpha^2\pi_\alpha(F)$. Consequently, $F=\prod_{\alpha\in I}u_\alpha^1\pi_\alpha\Big(\prod_{\alpha\in I}u_\alpha^2\pi_\alpha(F)\Big)$. Hence, F is a closed subset of $\prod_{\alpha\in I}(X_\alpha,u_\alpha^1,u_\alpha^2)$. □

3 CLOSED MAPS

Definition 3.1. *Let (X,u_1,u_2) and (Y,v_1,v_2) be biČech closure spaces. A map $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ is said to be* closed *(resp.* open*) if $f(F)$ is a closed (resp. open) subset of (Y,v_1,v_2) whenever F is a closed (resp. open) subset of (X,u_1,u_2).*

The following statement is evident :

Proposition 3.2. *Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biČech closure spaces. If $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ and $g:(Y,v_1,v_2)\to(Z,w_1,w_2)$ are closed (resp. open), then $g\circ f:(X,u_1,u_2)\to(Z,w_1,w_2)$ is closed (resp. open).*

Proposition 3.3. *Let $\{(X_\alpha,u_\alpha^1,u_\alpha^2):\alpha\in I\}$ be a family of biČech closure spaces. Then for each $\beta\in I$, the projection map $\pi_\beta:\prod_{\alpha\in I}(X_\alpha,u_\alpha^1,u_\alpha^2)\to(X_\beta,u_\beta^1,u_\beta^2)$ is closed.*

Proof. Let F be a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Then F is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ and $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$, respectively. Since $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u^1_\alpha) \to (X_\beta, u^1_\beta)$ is closed, $\pi_\beta(F)$ is a closed subset of (X_β, u^1_β). Similarly, since $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u^2_\alpha) \to (X_\beta, u^2_\beta)$ is closed, $\pi_\beta(F)$ is a closed subset of (X_β, u^2_β). Consequently, $\pi_\beta(F)$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Hence, π_β is closed. □

Proposition 3.4. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biČech closure spaces and let $\beta \in I$. Then F is a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Proof. Let $\beta \in I$ and let F be a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then F is a closed subset of (X_β, u^1_β) and (X_β, u^2_β), respectively. Since $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u^1_\alpha) \to (X_\beta, u^1_\beta)$ is continuous, $\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$. Similarly, since $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u^2_\alpha) \to (X_\beta, u^2_\beta)$ is continuous, $\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$. Consequently, $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ be a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Then $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ and $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$, respectively. Since $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u^1_\alpha) \to (X_\beta, u^1_\beta)$ is closed, $\pi_\beta\Big(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u^1_β). Similarly, since $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u^2_\alpha) \to (X_\beta, u^2_\beta)$ is closed, $\pi_\beta\Big(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u^2_β). Consequently, F is a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. □

Proposition 3.5. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biČech closure spaces and let $\beta \in I$. Then G is an open subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Proof. Let $\beta \in I$ and let G be an open subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then $X_\beta - G$ is a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. By Proposition 3.4, $(X_\beta - G) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. But $(X_\beta - G) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha = \prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$, hence $\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Therefore, $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ be an open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Then $\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. But $\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha = (X_\beta - G) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$, hence $(X_\beta - G) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. By Proposition 3.4, $X_\beta - G$ is a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Consequently, G is an open subset of $(X_\beta, u^1_\beta, u^2_\beta)$. □

Proposition 3.6. *Let (X, u_1, u_2) be a biČech closure space, $\{(Y_\alpha, v^1_\alpha, v^2_\alpha) : \alpha \in I\}$ be a family of biČech closure spaces and $f : X \to \prod_{\alpha\in I} Y_\alpha$ be a map. Then $f : (X, u_1, u_2) \to \prod_{\alpha\in I}(Y_\alpha, v^1_\alpha, v^2_\alpha)$is closed if and only if $\pi_\alpha \circ f : (X, u_1, u_2) \to (Y_\alpha, v^1_\alpha, v^2_\alpha)$is closed for each $\alpha \in I$.*

Proof. Let f be closed. Since π_α is closed for each $\alpha \in I$, also $\pi_\alpha \circ f$ is closed for each $\alpha \in I$.

Conversely, let $\pi_\alpha \circ f$ be closed for each $\alpha \in I$. Suppose that f is not closed. Then there exist a closed subset F of (X, u_1, u_2) such that

$$\prod_{\alpha\in I} v^1_\alpha \pi_\alpha\Big(\prod_{\alpha\in I} v^2_\alpha \pi_\alpha(f(F))\Big) \nsubseteq f(F).$$

Therefore, there exist $\beta \in I$ such that $v^1_\beta v^2_\beta \pi_\beta(f(F)) \nsubseteq \pi_\beta(f(F))$. But $\pi_\beta \circ f$ is closed, $\pi_\beta(f(F))$ is a closed subset of $(Y_\beta, v^1_\beta, v^2_\beta)$. This is a contradiction. □

Proposition 3.7. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ and $\{(Y_\alpha, v^1_\alpha, v^2_\alpha) : \alpha \in I\}$ be families of biČech closure spaces. For each $\alpha \in I$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a surjection and let $f : \prod_{\alpha\in I} X_\alpha \to \prod_{\alpha\in I} Y_\alpha$ be defined by $f\big((x_\alpha)_{\alpha\in I}\big) = \big(f_\alpha(x_\alpha)\big)_{\alpha\in I}$. Then $f : \prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha) \to \prod_{\alpha\in I}(Y_\alpha, v^1_\alpha, v^2_\alpha)$ is closed if and only if $f_\alpha : (X_\alpha, u^1_\alpha, u^2_\alpha) \to (Y_\alpha, v^1_\alpha, v^2_\alpha)$ is closed for each $\alpha \in I$.*

Proof. Let $\beta \in I$ and let F be a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Since f is closed, $f\Big(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big)$ is a closed subset of $\prod_{\alpha\in I}(Y_\alpha, v^1_\alpha, v^2_\alpha)$. But $f\Big(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) = f_\beta(F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} Y_\alpha$, hence

$f_\beta(F) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} Y_\alpha$ is a closed subset of $\prod_{\alpha \in I} (Y_\alpha, v^1_\alpha, v^2_\alpha)$. By Proposition 3.4, $f_\beta(F)$ is a closed subset of $(Y_\beta, v^1_\beta, v^2_\beta)$. Hence, f_β is closed.

Conversely, let f_β be closed for each $\beta \in I$. Suppose that f is not closed. Then there exist a closed subset F of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$ such that

$$\prod_{\alpha \in I} v^1_\alpha \pi_\alpha \Big(\prod_{\alpha \in I} v^2_\alpha \pi_\alpha(f(F)) \Big) \nsubseteq f(F).$$

Therefore, there exist $\beta \in I$ such that $v^1_\beta v^2_\beta \pi_\beta(f(F)) \nsubseteq \pi_\beta(f(F))$. But $\pi_\beta(F)$ is a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$ and f_β is closed, $f_\beta(\pi_\beta(F))$ is a closed subset of $(Y_\beta, v^1_\beta, v^2_\beta)$. This is a contradiction. □

Proposition 3.8. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ and $\{(Y_\alpha, v^1_\alpha, v^2_\alpha) : \alpha \in I\}$ be families of biČech closure spaces. For each $\alpha \in I$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a surjection and let $f : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ be defined by $f\big((x_\alpha)_{\alpha \in I}\big) = \big(f_\alpha(x_\alpha)\big)_{\alpha \in I}$. If $f : \prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha) \to \prod_{\alpha \in I} (Y_\alpha, v^1_\alpha, v^2_\alpha)$ is open, then $f_\alpha : (X_\alpha, u^1_\alpha, u^2_\alpha) \to (Y_\alpha, v^1_\alpha, v^2_\alpha)$ is open for each $\alpha \in I$.*

Proof. Let $\beta \in I$ and let G be an open subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is an open subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$. Since f is open, $f\Big(G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha\Big)$ is an open subset of $\prod_{\alpha \in I} (Y_\alpha, v^1_\alpha, v^2_\alpha)$. But $f\Big(G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha\Big) = f_\beta(G) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} Y_\alpha$, hence $f_\beta(G) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} Y_\alpha$ is an open subset of $\prod_{\alpha \in I} (Y_\alpha, v^1_\alpha, v^2_\alpha)$. By Proposition 3.5, $f_\beta(G)$ is an open subset of $(Y_\beta, v^1_\beta, v^2_\beta)$. Hence, f_β is open. □

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