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On Some Topological Properties of Semi-Metric Spaces Related to Fixed-Point Theory

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Abstract

In [4] S.-H. Cho, G.-Y. Lee, J.-S. Bae introduce (CC) property of convergence and proved some general coincidence and common fixed point theorems for self-mappings of semi-metric space which satisfies this property. Now we present some further results for this class of semi-metric spaces. We also give some open problems.

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1 Introduction and Preliminary Notes

There have been a number of generalizations of metric space. One such generalization is semi-metric space initiated by M. Fréchet, K. Menger [5], E. W. Chittenden and W. A. Wilson [7]. For references and historical remarks see [7]. For recent results see [3],[4].

Let X be a non-empty set and $d : X^2 \to [0, \infty)$. (X, d) is semi-metric space (symmetric space) if and only if it satisfies:

(W.1) $d(x, y) = 0$ if and only if $x = y$;

(W.2) $d(x, y) = d(y, x)$ if and only if $x = y$ for any $x, y \in X$.

Let (X, d) be a semi-metric (symmetric) space, $r > 0$ and $x \in X$, let $B(x, r) = \{y \in X : \ d(x, y) < r\}$. Let τ be the weakest topology on X such that the family $\{B(x, r) : \ x \in X, \ r \in [0, \infty)\}$ is the base for τ. Note that for every $\{x_n\} \subseteq X$ and $x \in X$, $\lim d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology τ.

Let (X, d) be a semi-metric (symmetric) space. Then:

- (X, d) satisfies the property (W3) if and only if from $\lim d(x_n, x) = 0$ and $\lim d(x_n, y) = 0$ follows $x = y$;
- (X, d) satisfies the property (W4) if and only if from $\lim d(x_n, x) = 0$ and $\lim d(x_n, y_n) = 0$ follows $\lim d(y_n, x) = 0$;
- (X, d) satisfies the property (HE) if and only if from $\lim d(x_n, x) = 0$ and $\lim d(y_n, x) = 0$ follows $\lim d(x_n, y_n) = 0$;
- (X, d) satisfies the property (CC) if and only if from $\lim d(x_n, x) = 0$ follows $\lim d(x_n, y) = d(x, y)$.
- (X, d) satisfies the property (W) if and only if from $\lim d(x_n, y_n) = 0$ and $\lim d(y_n, z_n) = 0$ follows $\lim d(x_n, z_n) = 0$.

Each of these conditions can be used as partial replacement for triangle inequality. (W3), (W4) was introduced by Wilson [7], (HE) by M. Aamri and D. El Moutawakil [1], (CC) by S.-H. Cho, G.-Y. Lee and J.-S. Bae [4] and (W) by D. Miheţ [6]. Note that $(W) \Rightarrow (W4) \Rightarrow (W3)$ (see [6]), $(W) \Rightarrow (HE)$ and $(CC) \Rightarrow (W3)$ (see [4]). In [4] authors present examples for following relationships: $(W3) \not\Rightarrow (W4)$, $(W4) \not\Rightarrow (HE)$, $(W4) \not\Rightarrow (CC)$, $(W3) \not\Rightarrow (HE)$, $(W3) \not\Rightarrow (CC)$, $(CC) \not\Rightarrow (W4)$, $(HE) \not\Rightarrow (CC)$, $(HE) \not\Rightarrow (W3)$, $(HE) \not\Rightarrow (W4)$ and $(CC) \not\Rightarrow (HE)$.

By $\mathcal{F}$ denote the set of all continuous, monotone nondecreasing, real functions $F : [0, \infty) \to [0, \infty)$ such that $F(x) = 0$ if and only if $x = 0$. In [8] it was proved:

Lemma 1.1 *(X. Zhang [8]). Let $F \in \mathcal{F}$ and $\{\varepsilon_n\} \subseteq [0, \infty)$, then from $F(\varepsilon_n) \to 0$ follows $\varepsilon_n \to 0$.*

In [3] we proved following results:

Theorem 1.2 *(I. Aranđelović, D. Petković [3]) Let (X, d) be a semi-metric space, $x \in X$, $\{x_n\} \subseteq X$ and $F \in \mathcal{F}$. Define $d^* : X^2 \to [0, \infty)$ by*

$$d^*(x, y) = F(d(x, y)), \text{ for any } x, y \in X.$$

Then:
1) (X, d^*) is semi-metric space;
2) $\lim d(x_n, x) = 0$ if and only if $\lim d^*(x_n, x) = 0$.
3) (X, d) satisfies the property (W3) if and only if (X^*, d) satisfies this property;
- (X, d) satisfies the property (W4) if and only if (X^*, d) satisfies this property;
- (X, d) satisfies the property (HE) if and only if (X^*, d) satisfies this property;
- (X, d) satisfies the property (W) if and only if (X^*, d) satisfies this property;
- (X, d) is complete if and only if (X^*, d) is complete.

As applicance of this results in [3] we proved that two common fixed point theorems from [2] follows from earlier results of M. Aamri, D. El Moutawakil (see [3]).

2 Results

Now we present some further results on property (CC).

Theorem 2.1 *Let (X, d) be a semi-metric space and $F \in \mathcal{F}$. Define $d^* : X^2 \to [0, \infty)$ by*

$$d^*(x, y) = F(d(x, y)),$$

for any $x, y \in X$. Then (X, d) satisfies the property (CC) if and only if (X^*, d) satisfies this property.

Proof. Let (X, d) be a semi-metric which satisfies the property (CC). Let $\lim F(d(x_n, x)) = 0$. By Lemma 1.1 it follows that $\lim d(x_n, x) = 0$ which implies $\lim d(x_n, y) = d(x, y)$. So

$$\lim F(d(x_n, y)) = F(\lim d(x_n, y)) = F(d(x, y)),$$

because F is continuous.

If (X, d^*) satisfies (CC), then from $\lim d(x_n, x) = 0$ it follows that

$$\lim F(d(x_n, x)) = F(\lim d(x_n, x)) = F(0) = 0,$$

because F is continuous. So $\lim F(d(x_n, y)) = d(x, y)$, because (X, d^*) satisfies (CC).$\diamondsuit$

Corollary 2.2 *Let (X, d) be a metric space and $F \in \mathcal{F}$. Define $d^* : X^2 \to [0, \infty)$ by*

$$d^*(x, y) = F(d(x, y)),$$

for any $x, y \in X$. Then (X, d^*) is a semi-metric space which satisfies the property (CC).

Proof. Let (X, d) be a metric space and $F \in \mathcal{F}$. For any $x, y \in X$ and $\{x_n\} \subseteq X$ such that $\lim d(x_n, x) = 0$, we have

$$\lim d(x_n, y) = d(\lim(x_n, y) = d(x, y),$$

because d is continuous and $\lim x_n = x$. So (X, d) has property (CC). $\diamondsuit$

3 Some open problems

From $(CC) \not\Rightarrow (W4)$ and $(W) \Rightarrow (W4)$ follows that $(CC) \not\Rightarrow (W4)$. But the following problem is still open.

Problem 3.1 *Let (X,d) be a semi-metric space which satisfies (W) and (HE). Is it satisfying (CC)?*

Problem 3.2 *Let (X,d) be a semi-metric space which satisfies (W). Is it satisfying (CC)?*

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