International Mathematical Forum, 4, 2009, no. 43, 2149 - 2156

On Contractibility Properties of Families of Stable Polynomials

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Abstract

In this paper, the contractibility properties of families of stable polynomials are studied. It is shown that families of stable polynomials are not contractible, but they can be divided into two disjoint subsets each of which is contractible. Thus, simple connectivity property of these families follows from contractibility properties.

Mathematics Subject Classification: 93D05, 54E40

Keywords: Stable Polynomials, Hurwitz stability, Schur stability, Simply-connected, Contractibility

1 Introduction

Family of stable polynomials and matrices with their topological properties have been extensively investigated in the literature (see e.g., [2–6], [9] and [10]). The present paper, with an aim of strengthening these results in mind, investigates the problem in terms of the contractibility of the families of stable polynomials. The contractibility of stable polynomials is important from the point of view of the following applications: Given a family of polynomials $\mathcal{P}$ is the family stable if the boundary of $\mathcal{P}$ is stable? Is the family $\mathcal{P}$ completely unstable if its boundary is completely unstable? Is a box of polynomials completely unstable if lower order boundaries are unstable? We believe that the present paper offers some new results that will be useful in the answers of the mentioned questions.

Firstly, let us introduce necessary topological notions that will be needed below. A contractible metric (topological) space is such that it can be reduced to one of its points by a continuous deformation. In other words let X be a metric space, $Y \subset X$ and $x_0 \in Y$. Y is called contractible to the point $x_0 \in Y$, if there exists a continuous function $F : Y \times [0,1] \to Y$ such that

$$F(x,0) = x, \ F(x,1) = x_0$$

for all $x \in Y$ ([7], [8]). If Y is contractible to a point $x_0 \in Y$, then it is contractible to any other point of Y.

The set Y is called path-wise connected if for all $a, b \in Y$ there exists a continuous function $f : [0,1] \to Y$ such that $f(0) = a, \ f(1) = b$. Such a function is called a curve in Y connecting a and b. If the end points of a curve are equal it is called a closed curve.

Let Y be a path-wise connected subset in the metric space X. Then Y is called simply-connected if there exists $x_0 \in Y$ such that any closed curve $f : [0,1] \to Y$ with $f(0) = f(1) = x_0$ can be deformed to the constant curve at x_0 in the following sense: There exists a continuous function $F : [0,1] \times [0,1] \to Y$ such that $F(t,0) = f(t), \ F(t,1) = F(0,s) = F(1,s) = x_0$ for all $t, s \in [0,1]$. (This property also holds for any other point of Y)

It follows from these definitions that if Y is contractible then Y is path-wise connected and simply-connected. The converse is not true in general. For example, ball's surface is path-wise connected and simply-connected but it is not contractible.

Suppose that $\mathcal{D}$ is a simply-connected open set in the complex plane $\mathbb{C}$. An n-th degree polynomial $p(s)$ ($n \times n$ matrix A) is called $\mathcal{D}$-stable if all its roots (eigenvalues) lie in the region $\mathcal{D}$. If $\mathcal{D}$ is the open left half plane then a $\mathcal{D}$-stable polynomial (matrix) is called Hurwitz stable; if $\mathcal{D}$ is the open unit disc then a $\mathcal{D}$-stable polynomial (matrix) is called Schur stable.

Fam & Meditch [5] established the contractibility of the set of Schur stable monic polynomials. Later, Duan & Patton proved, the product decomposition of the set of Hurwitz stable matrices of order n, into two convex open cones which itself forms a simply-connected open cone with a vertex at the origin (see [4]). In the same year, Fleming et. al. [6] proved the family of Schur D-stable matrices is open, unbounded, path-wise connected, nonconvex set in the matrix space. To the best of the author's knowledge, there are no results available in the literature dealing with the contractibility properties of the set of Hurwitz stable and $\mathcal{D}$-stable polynomials.

Consider n-th degree polynomials with complex coefficients:

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0. \tag{1}$$

Denote the set of all n-th degree $\mathcal{D}$-stable polynomials $p(s)$ by $\mathcal{P}^{\mathcal{D}}$ i.e.,

$$\mathcal{P}^{\mathcal{D}} \ = \ \{p(s) : \ p(s) \text{ is an } n\text{-th degree } \mathcal{D}\text{-stable complex polynomial}\}.$$

In section 2 we show that the set $\mathcal{P}^{\mathcal{D}}$ is not contractible, but it can be divided into two disjoint subsets each of which is contractible. Further we show that the family of n-th degree $\mathcal{D}$-stable monic polynomials with complex coefficients is contractible. At the end of the section 2 we consider real polynomials. We establish that the family of $\mathcal{D}$-stable real polynomials is not contractible but it can be divided into two disjoint subsets each of which is contractible.

We now recall a well known theorem from complex analysis which we shall need.

Theorem 1.1 (Riemann Mapping Theorem, [1], page 212). *If $\mathcal{D} \subset \mathbb{C}$, $\mathcal{D} \neq \mathbb{C}$ is a simply-connected open set, then there is a conformal equivalence f, that maps $\mathcal{D}$ onto the open unit disc $\mathcal{B}$.*

We need only part of this important theorem, namely that any simply-connected open proper subset of $\mathbb{C}$ is homeomorphic to the open unit disc. This property is obviously true for $\mathbb{C}$ also, but for our purposes the case $\mathcal{D} = \mathbb{C}$ is uninteresting, because the property of being $\mathcal{D}$-stable becomes then void. Nevertheless, the topological type of $\mathcal{P}^{\mathcal{D}}$ is not affected by that and it is the same for all simply-connected open $\mathcal{D}$ as we shall see below.

2 Contractibility of Families of Stable Polynomials

Proposition 2.1. *Let $\mathcal{D}$ be a simply-connected open set in the complex plane. Then, the family $\mathcal{P}^{\mathcal{D}}$ of n-th degree $\mathcal{D}$-stable complex polynomials is homeomorphic to the space of all n-th degree complex polynomials $\mathcal{P}^{\mathbb{C}}$.*

Proof. Let

$$p(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = a_n(s - s_1)(s - s_2)\cdots(s - s_n)$$

be an n-th degree $\mathcal{D}$-stable polynomial with complex coefficients.

As remarked above, $\mathcal{D}$ is homeomorphic to $\mathbb{C}$ so that we can find a continuous, one-to-one, onto map $f : \mathcal{D} \to \mathbb{C}$ with continuous inverse $f^{-1} : \mathbb{C} \to \mathcal{D}$.

Let us define a map from $\mathcal{P}^{\mathcal{D}}$ to $\mathcal{P}^{\mathbb{C}}$ as

$$\begin{aligned} F &: \mathcal{P}^{\mathcal{D}} \to \mathcal{P}^{\mathbb{C}}, \\ F(p(s)) &= a_n[s - f(s_1)] \cdot [s - f(s_2)] \cdots [s - f(s_n)]. \end{aligned}$$

It is clear that F is one-one, onto and continuous with the continuous inverse

$$\begin{aligned} F^{-1} &: \mathcal{P}^{\mathbb{C}} \to \mathcal{P}^{\mathcal{D}}, \\ F^{-1}(\tilde{p}(s)) &= a_n[s - f^{-1}(\tilde{s}_1)] \cdot [s - f^{-1}(\tilde{s}_2)] \cdots [s - f^{-1}(\tilde{s}_n)] \end{aligned}$$

for $\tilde{p}(s) = a_n(s - \tilde{s}_1)(s - \tilde{s}_2)\cdots(s - \tilde{s}_n)$. □

Proposition 2.2. *The space $\mathcal{P}^{\mathbb{C}}$ of all n-th degree complex polynomials is not simply-connected (and thus not contractible).*

Proof. The only condition on the coefficients of a polynomial in $\mathcal{P}^{\mathbb{C}}$ is $a_n \neq 0$. Then $\mathcal{P}^{\mathbb{C}}$ is homeomorphic to the product space $(\mathbb{C} - \{0\}) \times \mathbb{C} \times \cdots \times \mathbb{C} = (\mathbb{C} - \{0\}) \times \mathbb{C}^n$. It is a standard fact of algebraic topology that this space (which is deformable to $\mathbb{C} - \{0\}$) is not simply-connected. □

Corollary 2.3. *Let $\mathcal{D}$ be a simply-connected open set in the complex plane. Then, the space $\mathcal{P}^{\mathcal{D}}$ of all n-th degree $\mathcal{D}$-stable complex polynomials is not simply-connected (and thus not contractible).*

It can however be expressed as a disjoint union of two contractible subspaces. We give below such a decomposition.

Consider the following disjoint subsets of $\mathcal{P}^{\mathcal{D}}$.

$$\begin{aligned} \mathcal{P}_1^{\mathcal{D}} &= \{p(s) : a_n \notin (-\infty, 0),\ p(s) \text{ is } \mathcal{D}\text{-stable complex polynomial}\}, \\ \mathcal{P}_2^{\mathcal{D}} &= \{p(s) : a_n \in (-\infty, 0),\ p(s) \text{ is } \mathcal{D}\text{-stable complex polynomial}\}. \end{aligned}$$

It is clear that $\mathcal{P}^{\mathcal{D}} = \mathcal{P}_1^{\mathcal{D}} \cup \mathcal{P}_2^{\mathcal{D}}$.

Proposition 2.4. *Let $\mathcal{D}$ be a simply-connected open set in the complex plane. Then for an arbitrary fixed $s_* \in \mathcal{D}$ the followings hold.*

1. *The polynomial family $\mathcal{P}_1^{\mathcal{D}}$ is contractible to $(s - s_*)^n$,*
2. *The polynomial family $\mathcal{P}_2^{\mathcal{D}}$ is contractible to $-(s - s_*)^n$.*

Proof. Let f be a homeomorphism that maps $\mathcal{D}$ onto the unit disc $\mathcal{B}$. Consider a fixed n-th degree $\mathcal{D}$-stable polynomial with complex coefficients

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0,\ (a_n \neq 0).$$

It can be rewritten as $p(s) = a_n (s - s_1)(s - s_2) \cdots (s - s_n)$, where $s_i \in \mathcal{D}$ $(i = 1, 2, \ldots, n)$.

1. Define the polynomial

$$\begin{aligned} p_t(s) &= a_n \left[s - f^{-1}\left[(1-t) f(s_1) + t f(s_*)\right]\right] \\ &\quad \cdot \left[s - f^{-1}\left[(1-t) f(s_2) + t f(s_*)\right]\right] \cdots \\ &\quad \cdot \left[s - f^{-1}\left[(1-t) f(s_n) + t f(s_*)\right]\right],\ t \in [0, 1] \end{aligned}$$

and the function

$$\begin{aligned} F &: \mathcal{P}_1^{\mathcal{D}} \times [0, 1] \to \mathcal{P}_1^{\mathcal{D}}, \\ F(p(s), t) &= \frac{p_t(s)}{(1-t) + t a_n},\ a_n \notin (-\infty, 0). \end{aligned}$$

From continuity of f and f^{-1} it follows that F is continuous. Moreover,

$$\begin{aligned} F(p(s),0) &= p(s), \\ F(p(s),1) &= (s-s_*)^n \end{aligned}$$

holds and $(s-s_*)^n \in \mathcal{P}_1^{\mathcal{D}}$. Hence, the polynomial family $\mathcal{P}_1^{\mathcal{D}}$ is contractible to $(s-s_*)^n$.

2. Similarly, the function

$$\begin{aligned} \widetilde{F} &: \mathcal{P}_2^{\mathcal{D}} \times [0,1] \to \mathcal{P}_2^{\mathcal{D}}, \\ \widetilde{F}(p(s),t) &= \frac{p_t(s)}{(1-t)-ta_n}, \; a_n \in (-\infty, 0) \end{aligned}$$

is also continuous and

$$\begin{aligned} \widetilde{F}(p(s),0) &= p(s), \\ \widetilde{F}(p(s),1) &= -(s-s_*)^n \end{aligned}$$

holds. Therefore, the family $\mathcal{P}_2^{\mathcal{D}}$ is contractible to the polynomial $-(s-s_*)^n \in \mathcal{P}_2^{\mathcal{D}}$. This completes the proof.

□

Now we consider monic polynomials. Denote

$$\mathcal{M}^{\mathcal{D}} = \{p(s) : p(s) \text{ is an } n\text{-th degree } \mathcal{D}\text{-stable complex monic polynomial}\}.$$

Proposition 2.5. *Let $\mathcal{D}$ be a simply-connected open set in the complex plane and $s_* \in \mathcal{D}$ be an arbitrary fixed point. Then the family $\mathcal{M}^{\mathcal{D}}$ is contractible to the polynomial $(s-s_*)^n$.*

Proof. Let $f : \mathcal{D} \to \mathcal{B}$ be a homeomorphism. For $p(s) \in \mathcal{M}^{\mathcal{D}}$, $p(s) = (s-s_1)(s-s_2)\cdots(s-s_n)$ define the function

$$\begin{aligned} F &: \mathcal{M}^{\mathcal{D}} \times [0,1] \to \mathcal{M}^{\mathcal{D}}, \\ F(p(s),t) &= \left[s - f^{-1}\left[(1-t)f(s_1) + tf(s_*)\right]\right] \\ &\quad \cdot \left[s - f^{-1}\left[(1-t)f(s_2) + tf(s_*)\right]\right] \cdots \\ &\quad \cdot \left[s - f^{-1}\left[(1-t)f(s_n) + tf(s_*)\right]\right]. \end{aligned}$$

This function is continuous and

$$F(p(s),0) = p(s), \; F(p(s),1) = (s-s_*)^n.$$

□

Now we consider real polynomials. Assume that $\mathcal{D} = \mathcal{D}_1 \cup \overline{\mathcal{D}}_1$, where $\mathcal{D}_1$ is a simply-connected open set in the complex plane and $\overline{\mathcal{D}}_1$ denotes complex conjugate of $\mathcal{D}_1$. Obviously the whole set of $\mathcal{D}$-stable n-th degree real polynomial family is not path-wise connected (consequently is not simply-connected): If $p(s)$ is $\mathcal{D}$-stable real polynomial with $a_n < 0$ then there is not a continuous $\mathcal{D}$-stable path connecting $p(s)$ and $-p(s)$ and consisting of n-th degree real polynomials.

Define the disjoint sets

$$\begin{aligned} \mathcal{R}_1^{\mathcal{D}} &= \{p(s) : a_n > 0,\ p(s) \text{ is an } n\text{-th degree } \mathcal{D}\text{-stable real polynomial}\}, \\ \mathcal{R}_2^{\mathcal{D}} &= \{p(s) : a_n < 0,\ p(s) \text{ is an } n\text{-th degree } \mathcal{D}\text{-stable real polynomial}\}. \end{aligned}$$

If n is odd and $\mathcal{D}$ does not intersect the real axis then $\mathcal{R}_1^{\mathcal{D}}$ and $\mathcal{R}_2^{\mathcal{D}}$ are empty. Therefore for odd n we assume that $\mathcal{D}$ intersects the real axis $\mathbb{R}$. Under this assumption the sets $\mathcal{R}_1^{\mathcal{D}}$, $\mathcal{R}_2^{\mathcal{D}}$ are nonempty. Indeed, if n is odd $(n = 2k + 1)$ then for $s_* \in \mathcal{D} \cap \mathbb{R}$

$$p_*(s) = (s - s_*)^n \in \mathcal{R}_1^{\mathcal{D}}, \quad -p_*(s) \in \mathcal{R}_2^{\mathcal{D}}.$$

If n is even $(n = 2k)$ then for $z_* \in \mathcal{D}$

$$p_*(s) = (s - z_*)^k (s - \overline{z}_*)^k \in \mathcal{R}_1^{\mathcal{D}}, \quad -p_*(s) \in \mathcal{R}_2^{\mathcal{D}}.$$

Proposition 2.6. *Assume that $n = 2k$, $\mathcal{D} = \mathcal{D}_1 \cup \overline{\mathcal{D}}_1$, where $\mathcal{D}_1$ is a simply-connected open set in the complex plane. Let $s_* \in \mathcal{D}_1$ be an arbitrary fixed complex number. Then the set $\mathcal{R}_1^{\mathcal{D}}$ is contractible to $[(s - s_*)(s - \overline{s}_*)]^k$ and the set $\mathcal{R}_2^{\mathcal{D}}$ is contractible to $-[(s - s_*)(s - \overline{s}_*)]^k$.*

Proof. Choose again a homeomorphism $f : \mathcal{D}_1 \to \mathcal{B}$.

For the polynomial $p(s) = a_n(s - s_1) \cdots (s - s_k)(s - \overline{s}_1) \cdots (s - \overline{s}_k)$ and $t \in [0, 1]$ define

$$\begin{aligned} p_t(s) = \; & a_n \left[s - f^{-1}\left[(1-t)f(s_1) + tf(s_*)\right]\right] \cdots \\ & \cdot \left[s - f^{-1}\left[(1-t)f(s_k) + tf(s_*)\right]\right] \\ & \cdot \left[s - \overline{f^{-1}\left[(1-t)f(s_1) + tf(s_*)\right]}\right] \cdots \\ & \cdot \left[s - \overline{f^{-1}\left[(1-t)f(s_k) + tf(s_*)\right]}\right]. \end{aligned}$$

It is clear that the function

$$\begin{aligned} F &: \mathcal{R}_1^{\mathcal{D}} \times [0, 1] \to \mathcal{R}_1^{\mathcal{D}}, \\ F(p(s), t) &= \frac{p_t(s)}{(1-t) + ta_n} \end{aligned}$$

is continuous (here $a_n > 0$). Additionally,

$$F(p(s),0) = p(s),\ F(p(s),1) = [(s - s_*) \cdot (s - \overline{s}_*)]^k .$$

Similarly, the function

$$\begin{aligned} F &: \mathcal{R}_2^{\mathcal{D}} \times [0,1] \to \mathcal{R}_2^{\mathcal{D}}, \\ F(p(s),t) &= \frac{p_t(s)}{(1-t) - ta_n} \end{aligned}$$

is also continuous (here $a_n < 0$) and

$$F(p(s),0) = p(s),\ F(p(s),1) = -[(s - s_*) \cdot (s - \overline{s}_*)]^k .$$

□

Proposition 2.7. *Assume that n is odd, $\mathcal{D}$ is a simply-connected open set in the complex plane, symmetric with respect to the real axis and intersects the real axis. Let $s_* \in \mathcal{D} \cap \mathbb{R}$ be an arbitrary fixed number. Then the set $\mathcal{R}_1^{\mathcal{D}}$ is contractible to $(s - s_*)^n$ and the set $\mathcal{R}_2^{\mathcal{D}}$ is contractible to $-(s - s_*)^n$.*

Proof. Let the function $f : \mathcal{D} \to \mathcal{B}$ be a homeomorphism onto the unit disk. For $p(s) = a_n(s - s_1)(s - s_2) \cdots (s - s_n)$ and $t \in [0,1]$ define the polynomial

$$p_t(s) \ = \ a_n[s - f^{-1}[(1-t)f(s_1) + tf(s_*)]] \cdots [s - f^{-1}[(1-t)f(s_n) + tf(s_*)]].$$

and the continuous function

$$\begin{aligned} F &: \mathcal{R}_1^{\mathcal{D}} \times [0,1] \to \mathcal{R}_1^{\mathcal{D}}, \\ F(p(s),t) &= \frac{p_t(s)}{(1-t) + ta_n} \end{aligned}$$

(here $a_n > 0$). Then

$$F(p(s),0) = p(s),\ F(p(s),1) = (s - s_*)^n.$$

In the same way, the function

$$\begin{aligned} \widetilde{F} &: \mathcal{R}_2^{\mathcal{D}} \times [0,1] \to \mathcal{R}_2^{\mathcal{D}}, \\ \widetilde{F}(p(s),t) &= \frac{p_t(s)}{(1-t) - ta_n} \end{aligned}$$

(here $a_n < 0$) is also continuous and

$$\widetilde{F}(p(s),0) = p(s),\ \widetilde{F}(p(s),1) = -(s - s_*)^n.$$

□

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Received: April, 2009