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An Affine Space over a Module

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Abstract. In the paper basic notions of an affine space over a linear space, usually called an affine space over a field, have been made generalizations in an affine space over a module. Some properties of the affine mapping in an affine space over a module are also investigated. Isomorphism of affine spaces associated with a module is considered as well.

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1. Introduction

By taking more general point of view, one can generalize the theory of affine spaces over a field and consider an affine space over a module. Note at first that after Bourbaki [2] by a ring we understand an associative unitary ring and in the paper is always assumed that 3 is an invertible element. To establish notation recall now the concept of a module.

Definition. Let P be a ring, and let (M, +) be an abelian group. Then M is called a left P-module if there exists a scalar multiplication $P \times M \to M$, such that for all $\alpha, \beta, 1 \in P$ and all $u, v \in M$, the following axioms are fulfilled:

M1. $\alpha(u + v) = \alpha u + \alpha v$,
M2. $(\alpha + \beta)u = \alpha u + \beta u$,
M3. $\alpha(\beta u) = (\alpha\beta)u$,
M4. $1u = u$.

The following corollary, due to Artin [1], states an interesting and not commonly known fact, namely the assumption that (M, +) is an abelian group is too strong. In fact the abelianity condition can be omitted.

Corollary (Artin). Axioms M1–M4 imply that the group (M, +) is abelian.

P r o o f. From one hand, we have

$$(1 + 1)(u + v) = (u + v) + (u + v) = u + (v + u) + v.$$

On the other side, we obtain

$$(1 + 1)(u + v) = (1 + 1)u + (1 + 1)v = u + (u + v) + v.$$

Therefore

$$u + v = v = u.$$ ■

2. An affine space over a module

The concept of an affine space over a module (or an affine space associated with a module) was introduced by Bourbaki in [2].

Definition 1. An affine space over a module is called an algebraic structure of the form (A, M_A, ω), where A is a nonempty set (elements of which are called the points of the affine space), M_A stands for a module associated to the set A, and $\omega: A \times A \to M_A$ is a mapping that for any points $a, b, c \in A$ the following axioms hold:

A_1. Every triple of points $a, b, c \in A$ satisfies the relation

$$\omega(a, b) + \omega(b, c) + \omega(c, a) = 0, \tag{1}$$

A_2. The mapping $\chi_a(x) = \omega(x , a)$ is a bijection; $x \in A$. (2)

The relation (1) is called Chasle's equality. The dimension of M_A is taken for dimention of the affine space. If $\dim M_A = n$, then (A, M_A, ω) is called an n-dimentional affine space over a module M_A, and n-tuples of the module M_A are called vectors. It is said that the module M_A is tangent to the given affine space.

The mapping χ_a is called the mapping fixed at the point $a \in A$, and the set of all mappings, i.e. $\{\chi_a\colon a \in A\}$, is called the atlas of the given affine space. The following theorem finds some properties of the mappings ω and χ_a.

Theorem 1. If $a, b \in A$ and $u \in M_A$, then

$$\text{(i) } \omega(a, a) = 0, \tag{3}$$

$$\text{(ii) } \omega(a, b) = -\,\omega(b, a), \tag{4}$$

$$\text{(iii) } (\chi_b \circ \chi_a^{-1})(u) = u + \omega(a, b). \tag{5}$$

P r o o f. (i): If we set a = b = c in (1), then we obtain

$$3\omega(a, a) = 0.$$

Since 3 is invertible by assumption, therefore (i) holds true.

(ii): Setting $c = a$ in (1) we have

$$\omega(a, b) + \omega(b, a) + \omega(a, a) = 0.$$

From (i), we conclude that (ii).

(iii): Note that (2) implies $\chi_a(c) = u$, then $\chi_a^{-1}(u) = c$. Moreover, $\omega(\chi_a^{-1}(u), a) = u$. By (1) and (4), for points a, b and $\chi_a^{-1}(u)$, we have step by step as follows

$$\begin{aligned}(\chi_b \circ \chi_a^{-1})(u) &= \chi_b(\chi_a^{-1}(u)) = \omega(\chi_a^{-1}(u), b) = \omega(\chi_a^{-1}(u), b) + 0 = \\ &= \omega(\chi_a^{-1}(u), b) + \omega(b, \chi_a^{-1}(u)) + \omega(\chi_a^{-1}(u), a) + \omega(a, b) = \\ &= 0 + \omega(\chi_a^{-1}(u), a) + \omega(a, b) = \\ &= u + \omega(a, b).\end{aligned}$$ ■

Definition 2. The affine addition and subtraction of points $a \in A$ and vectors $u \in M_A$ is defined in the following way:

$$a + u = \chi_a^{-1}(u) \in A, \tag{6}$$
$$a - u = \chi_a^{-1}(-u) \in A. \tag{7}$$

Corollary 1. Since by axiom A_2 any affine mapping χ_a is a bijection, therefore both addition and subtraction are bijections for all fixed points $a \in A$, i.e.

$$\chi_a^{-1} \circ \chi_a = id_A,\ \chi_a \circ \chi_a^{-1} = id_{M_A}. \tag{8}$$

In the following theorem is said about some basic properties of affine addition and subtraction.

Theorem 2. If $a, b \in A$ and $u, v \in M_A$, then

$$\text{(i)}\ \ a + \omega(b, a) = b, \tag{9}$$
$$\text{(ii)}\ \ (a + u) + v = a + (u + v), \tag{10}$$
$$\text{(iii)}\ \ \omega(a + u, b) = u + \omega(a, b) = \omega(a, b - u). \tag{11}$$

P r o o f. (i): By (1), (2), and from (8), we have

$$a + \omega(b, a) = \chi_a^{-1}(\omega(b, a)) = \chi_a^{-1}(\chi_a(b)) = id_A(b) = b.$$

(ii): Let set $u = \omega(b, a)$, $v = \omega(c, b)$, where $a, b, c \in A$. Therefore by (i) of the given theorem we have

$$(a + u) + v = [a + \omega(b, a)] + \omega(c, b) = b + \omega(c, b) = c.$$

On the other hand

$$a + (u + v) = a + [\omega(b, a) + \omega(c, b)] = a + \omega(c, a) = c,$$

and that ends the proof of (ii).

(iii): To prove the property (iii) it is enough to show that both $\omega(a + u, b)$ and $\omega(a, b - u)$ are equal to $u + \omega(a, b)$. Really,

$$\omega(a + u, b) = \chi_b(a + u) = (\chi_b \circ \chi_a)(u) = u + \omega(a, b),$$

and

$$\begin{aligned}\omega(a, b - u) &= -\,\omega(b - u, a) = -\,\chi_a \circ \chi_b^{-1}(-u) = \\ &= -[-u + \omega(b, a)] = u + \omega(a, b).\end{aligned}$$ ■

Corollary 2. In an affine space over a module the following properties hold:

$$a + u = b \iff u = \omega(b, a), \tag{12}$$
$$\omega(a + u, b + u) = \omega(a, b), \tag{13}$$
$$\omega(a + u, a + v) = u - v, \tag{14}$$
$$\omega(a, b) = \omega(c, d) \iff \omega(a, c) = \omega(b, d), \tag{15}$$
$$a + u = b + u \implies a = b, \tag{16}$$
$$a + u = a + v \implies u = v. \tag{17}$$

P r o o f. (12): From (6), and since χ_a is a bijection, we obtain

$\Rightarrow$: $a + u = b \implies \chi_a^{-1}(u) = b \implies (\chi_a \circ \chi_a^{-1})(u) = \chi_a(b) \implies u = \omega(b, a)$.

$\Leftarrow$: $u = \omega(b, a) \implies u = \chi_a(b) \implies \chi_a^{-1}(u) = (\chi_a^{-1} \circ \chi_a)(b) \implies a + u = b$.

(13): Let $u = \chi_b(c)$ for some $c\in A$. Therefore from (11) and Chasle's equality we have

$$\omega(a + u, b + u) = u + \omega(a, b + u) = \chi_b(c) + \omega(a, \chi_b^{-1}(u)) =$$
$$= \omega(c, b) + \omega(a, (\chi_b^{-1} \circ \chi_b)(c)) = \omega(c, b) + \omega(a, c) = \omega(a, b).$$

(14) By (3), and (11) we obtain

$$\omega(a + u, a + v) = u + \omega(a, a + v) = u - \omega(a + v, a) =$$
$$= u - v - \omega(a, a) = u - v.$$

(15) $\Rightarrow$: $\omega(a, b) = \omega(c, d) \iff \omega(a, b) + \omega(d, c) = 0 \iff$

$$\iff \omega(a, b) + \omega(b, c) + \omega(c, b) + \omega(d, c) = 0 \iff$$
$$\iff -\omega(c, a) - \omega(b, d) = 0 \iff \omega(a, c) = \omega(b, d).$$

$\Leftarrow$: The proof runs in the same way.

It easy can be seen that since χ_a is a bijective mapping, then the properties (16) and (17) hold true as well. ■

Example 1. An affine space over a linear space is the affine space over the module.

Example 2. Let M be a unitary module, where the function ω: $M \times M \to M$ is defined as follows:

$$\omega(u, v) = u - v;\ u, v\in M.$$

The algebraic structure (M, M, ω) is an affine space over a module. In particular case, if $M = R^n$, where R is a set of real numbers, then the affine space (R^n, R^n, ω) is said to be the standard affine space ([4], Chapter 1).

3. Affine Mapping and Isomorphism

Let start from the following definition of an affine mapping of two affine spaces, not necessarily associated with the same module. It is only assumed here that modules are over the same ring.

Definition 3. Let (A, M_A, α) and (B, M_B, β) be affine spaces over a module. A mapping σ: $A \to B$ is called affine if there exists a linear mapping φ: $M_A \to M_B$ satisfying the following condition

$$\beta(\sigma(a), \sigma(b)) = \varphi(\alpha(a, b));\ a, b\in A \tag{18}$$

or equivalently

$$(\chi_{\sigma(b)} \circ \sigma)(a) = (\varphi \circ \chi_b)(a), \tag{19}$$

which can be presented as the following diagram

$$\begin{tikzcd} A \arrow[r, "\chi_b"] \arrow[d, "\sigma"'] & M_A \arrow[d, "\varphi"] \\ B \arrow[r, "\chi_{\sigma(b)}"] & M_B \end{tikzcd}$$

Corollary 3. Since χ_b is a bijection, therefore if there is an affine mapping given by (18), it is unique.

Remark. A generalization of an affine mapping was introduced by Vijayaraju and Marusadai in [5], and next a fixed-point theorem for generalized affine mapping was obtained by Nashine in [3].

Definition 4. For fixed $u \in M_A$ a mapping τ_u: $A \to A$ defined as an affine sum

$$\tau_u(a) = a + u \tag{20}$$

is called translation by u of an affine space.

Lemma 1. Let (A, M_A, ω) be an affine space associated with a module. An affine mapping σ: $A \to A$ is translation iff a linear mapping φ: $M_A \to M_A$ is identity.

P r o o f. =>: By assumption and (18), for any $a, b \in A$ we have

$$\omega(\sigma(a), \sigma(b)) = \varphi(\omega(a, b)). \tag{21}$$

Since σ is translation, therefore by Definition 4 there exists a fixed $u \in M_A$ that for any $a \in A$ the equality $\sigma(a) = a + u$ is true, and (21) can be rewritten as

$$\omega(a + u, b + u) = \varphi(\omega(a, b)), \tag{22}$$

and by (13) we obtain

$$\omega(a, b) = \varphi(\omega(a, b)).$$

That means the mapping φ is identity on M_A.

<=: By (19), since $\varphi = id$, we have

$$\chi_{\sigma(b)} \circ \sigma = \varphi \circ \chi_b = \chi_b.$$

From above, setting $c = \sigma(b)$, we obtain

$$\sigma = \chi_c^{-1} \circ \chi_b.$$

Therefore, from (2) and (6), for any element $d \in A$ is

$$\sigma(d) = (\chi_c^{-1} \circ \chi_b)(d) = \chi_c^{-1}(\omega(d, b)) = c + \omega(d, b).$$ ■

Lemma 2. Let (A, M_A, α) and (B, M_B, β) be affine spaces over a module. If σ: $A \to B$ is an affine mapping, and a linear mapping φ: $M_A \to M_A$ satisfies (18), then for any element $a \in A$ the following relation is true

$$\sigma = \tau^{-1}_{\sigma(a)} \circ \varphi \circ \chi_a. \tag{23}$$

P r o o f. Let be given a fixed $a \in A$. Then for any $b \in A$, from (18) and (8), we obtain

$$\sigma(b) = (\tau^{-1}_{\sigma(a)} \circ \varphi \circ \chi_a)(b) = (\tau^{-1}_{\sigma(a)} \circ \varphi)(\alpha(a, b)) =$$
$$= (\tau^{-1}_{\sigma(a)})(\beta(\sigma(a), \sigma(b)) = (\tau^{-1}_{\sigma(a)} \circ \tau_{\sigma(a)})(\sigma(b) =$$
$$= id_B(\sigma(b)) = \sigma(b).$$

■

Definition 5. Let (A, M_A, α) and (B, M_B, β) be affine spaces over a module. An affine mapping $\sigma: A \to B$ is said to be an isomorphism of the given affine spaces if the corresponding mapping $\varphi \in Hom(M_A, M_B)$ is an isomorphism.

As it is known that very general concept of structure preserving map appears in large variety areas of mathematics. Recall that two affine spaces are called isomorphic if there exists an isomorphism between them. Fundamental is the following

Theorem 3. There exists unique (up to isomorphism) *n*-dimensional affine space over a module.

P r o o f. Let (A, M_A, ω) be *n*-dimensional affine space associated with a module M_A. To show that the space (A, M_A, ω) is isomorphic to (M^n, M^n, ω), where M is a module, we fix a point $a \in A$ and a base $\{u_1 , u_2, \ldots, u_n\} \subset M_A$. The base implies an isomorphism $I: M_A \to M^n$ such that

$$u = \sum_{i=1}^{n} \lambda_i u_i \quad \Rightarrow \quad I(u) = \begin{bmatrix} \lambda_1 \\ \ldots \\ \lambda_n \end{bmatrix} \in M^n. \tag{24}$$

Define the mapping $\sigma: A \to M^n$ as

$$\sigma = I \circ \chi_a. \tag{25}$$

By (25) and linearity of the mapping I defined in (24), we have

$$\sigma(b) - \sigma(c) = (I \circ \chi_a)(b) - (I \circ \chi_a)(c) = I(\chi_a(b) - \chi_a(c)); \ b, c \in A,$$

and on the other side

$$\chi_a(b) - \chi_a(c) = \omega(b, a) - \omega(c, a) = \omega(b, c).$$

Therefore

$$\sigma(b) - \sigma(c) = I(\omega(b, c)),$$

and it means that (18) is satisfied. ■

Corollary 4. Finite dimensional affine spaces over the same module are isomorphic iff they have the same dimension (see also [6]).

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