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A Subclass of Starlike Functions with Respect to Conjugate Points

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Abstract

Let $S_c^*(A,B)$ denote the class of functions f which are analytic in an open unit disc $\mathcal{D} = \{z : |z| < 1\}$ and satisfying the condition $\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}} \prec \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $z \in \mathcal{D}$. The aim of paper is to determine coefficient estimates for the class $S_c^*(A,B)$.

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1 Introduction

Let $\mathcal{U}$ be the class of functions which are analytic in the open unit disc $\mathcal{D} = \{z : |z| < 1\}$ given by

$$w(z) = \sum_{k=1}^{\infty} b_k z^k$$

and satisfying the conditions

$$w(0) = 0,\ |w(z)| < 1,\ z \in \mathcal{D}.$$

Let $\mathcal{S}$ denote the class of functions f which are analytic and univalent in $\mathcal{D}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \mathcal{D}. \quad (1)$$

Also, let $\mathcal{S}_s^*$ be the subclass of $\mathcal{S}$ consisting of functions given by (1) satisfying

$$Re \ \left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0, \quad z \in \mathcal{D}.$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi in 1959. El-Ashwah and Thomas in [1], introduced two other classes namely the class $\mathcal{S}_c^*$ consisting of functions starlike with respect to conjugate points and $\mathcal{S}_{sc}^*$ consisting of functions starlike with respect to symmetric conjugate points.

Further, let $f, g \in \mathcal{U}$. Then we say that f is subordinate to g, and we write $f \prec g$, if there exists a function $w \in \mathcal{U}$ such that $f(z) = g(w(z))$ for all $z \in \mathcal{D}$. Specially, if g is univalent in $\mathcal{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathcal{D}) \subseteq g(\mathcal{D})$.

In terms of subordination, Goel and Mehrok in 1982 introduced a subclass of $\mathcal{S}_s^*$ denoted by $\mathcal{S}_s^*(A,B)$. Let $\mathcal{S}_s^*(A,B)$ denote the class of functions of the form (1) and satisfying the condition

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1, \ z \in \mathcal{D}.$$

In this paper, let consider $\mathcal{S}_c^*(A,B)$ be the class of functions of the form (1) and satisfying the condition

$$\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, \ z \in \mathcal{D}.$$

Obviously $\mathcal{S}_c^*(A,B)$ is a subclass of the class $\mathcal{S}_c^* = \mathcal{S}_c^*(1,-1)$.

By definition of subordination it follows that $f \in \mathcal{S}_c^*(A,B)$ if and only if

$$\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} = \frac{1+Aw(z)}{1+Bw(z)} = P(z), \quad w \in \mathcal{U} \tag{2}$$

where

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{3}$$

We study the class $\mathcal{S}_c^*(A,B)$ and obtain coefficient estimates.

2 Preliminary Result

We need the following preliminary lemma, required for proving our result.

Lemma 2.1 *([2]) If $P(z)$ is given by (3) then*

$$|p_n| \leq (A - B). \tag{4}$$

3 Main Result

We give the coefficient inequalities for the class $S_c^*(A, B)$.

Theorem 3.1 *Let $f \in S_c^*(A, B)$, then for $n \geq 1$,*

$$|a_{2n}| \leq \frac{(A - B)}{(2n - 1)!} \prod_{j=1}^{2n-2} (A - B + j), \tag{5}$$

$$|a_{2n+1}| \leq \frac{(A - B)}{(2n)!} \prod_{j=1}^{2n-1} (A - B + j). \tag{6}$$

Proof.

For (2) and (3), we have

$$2(z + 2a_2z^2 + 3a_3z^3 + ... + 2na_{2n}z^{2n} + (2n + 1)a_{2n+1}z^{2n+1} + ...)$$

$$\begin{aligned} = \quad & 2(z + a_2z^2 + a_3z^3 + a_4z^4 + ... + a_{2n}z^{2n} + a_{2n+1}z^{2n+1} + ...) \\ & \bullet\ (1 + p_1z + p_2z^2 + ... + p_{2n}z^{2n} + p_{2n+1}z^{2n+1} + ...) \end{aligned}$$

Equating the coefficients of like powers of z, we have

$$a_2 = p_1, \quad 2a_3 = p_2 + a_2p_1 \tag{7}$$

$$3a_4 = p_3 + a_2p_2 + a_3p_1, \quad 4a_5 = p_4 + a_2p_3 + a_3p_2 + a_4p_1 \tag{8}$$

$$(2n - 1)a_{2n} = p_{2n-1} + a_2p_{2n-1} + a_3p_{2n-3} + ... + a_{2n-1}p_1 \tag{9}$$

$$(2n)a_{2n+1} = p_{2n} + a_2p_{2n-1} + a_3p_{2n-2} + ... + a_{2n}p_1. \tag{10}$$

Easily using Lemma 2.1 and (7), we get

$$|a_2| \leq A-B, \quad |a_3| \leq \frac{(A-B)(A-B+1)}{2}. \tag{11}$$

Again by applying (11) and followed by Lemma 2.1, we get from (8)

$$|a_4| \leq \frac{(A-B)(A-B+1)(A-B+2)}{2(3)}$$

$$|a_5| \leq \frac{(A-B)^4+6(A-B)^3+11(A-B)^2+6(A-B)}{2(3)(4)}.$$

It follows that (5) and (6) hold for n=1,2. We now prove (5) using induction. Equation (9) in conjuction with Lemma 2.1 yield

$$|a_{2n}| \leq \frac{A-B}{2n-1}\left[1+\sum_{k=1}^{n-1}|a_{2k}|+\sum_{k=1}^{n-1}|a_{2k+1}|.\right] \tag{12}$$

We assume that (5) holds for k=3,4,...,(n-1). Then from (12), we obtain

$$|a_{2n}| \leq \frac{A-B}{2n-1}\left[1+\sum_{k=1}^{n-1}\frac{A-B}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j)+\sum_{k=1}^{n-1}\frac{A-B}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right]. \tag{13}$$

In order to complete the proof, it is sufficient to show that

$$\frac{A-B}{2m-1}\left[1+\sum_{k=1}^{m-1}\frac{A-B}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j)+\sum_{k=1}^{m-1}\frac{A-B}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right]$$

$$= \frac{A-B}{(2m-1)!}\prod_{j=1}^{2m-2}(A-B+j), \quad (m=3,4,...,n). \tag{14}$$

(14) is valid for $m=3$.

Let us suppose that (14) is true for all m, $3<m\leq(n-1)$. Then from (13)

$$\frac{A-B}{2n-1}\left[1+\sum_{k=1}^{n-1}\frac{A-B}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j)+\sum_{k=1}^{n-1}\frac{A-B}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right]$$

$$= \left(\frac{2n-3}{2n-1}\right)\left(\frac{A-B}{2(n-1)-1}\left(1+\sum_{k=1}^{n-2}\frac{A-B}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j)+\sum_{k=1}^{n-2}\frac{A-B}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right)\right)$$

$$
\begin{aligned}
& + \frac{A-B}{2n-1}\,\frac{A-B}{(2(n-1)-1)!}\prod_{j=1}^{2n-4}(A-B+j) + \frac{A-B}{2n-1}\,\frac{A-B}{(2(n-1))!}\prod_{j=1}^{2n-3}(A-B+j) \\
= \; & \frac{2n-3}{2n-1}\,\frac{A-B}{(2(n-1)-1)!}\prod_{j=1}^{2n-4}(A-B+j) \\
& + \frac{A-B}{2n-1}\,\frac{A-B}{(2(n-1)-1)!}\prod_{j=1}^{2n-4}(A-B+j) + \frac{A-B}{2n-1}\,\frac{A-B}{(2(n-1))!}\prod_{j=1}^{2n-3}(A-B+j) \\
= \; & \frac{A-B}{(2n-1)(2(n-1)-1)!}\prod_{j=1}^{2n-4}(A-B+j)(A-B+2n-3) \\
& + \frac{A-B}{(2n-1)}\,\frac{A-B}{(2(n-1))!}\prod_{j=1}^{2n-3}(A-B+j) \\
= \; & \frac{A-B}{(2n-1)!}\prod_{j=1}^{2(n-1)}(A-B+j)
\end{aligned}
$$

Thus, (14) holds for $m = n$ and hence (5) follows. Similarly, we can prove (6).

References

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