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Viscosity Approximate Methods for a Finite Family of Asymptotically Pseudocontractive Mappings

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Abstract

Let K be a nonempty closed convex and bounded subset of a real Banach space E; Let $f : K \to K$ be a *contraction map with a contractive constant* $\alpha \in (0,1)$ and $T_i : K \to K$, $i = 1, 2, \cdots, N$, be N *uniformly L-Lipschitzian*, *asymptotically pseudocontractive maps with sequences* $\{k_n^{(i)}\}$, and *uniformly asymptotically regular with sequence* $\{\varepsilon_n\}$. Suppose $\{x_n\}$ is generated iteratively by $x_{n+1} := \lambda_n\theta_n f(x_n) + [1-\lambda_n(1+\theta_n)]x_n + \lambda_n T_n^n x_n$ $n \geq 1$, where $T_n = T_{n(mod\, N)}$, $x_1 \in K$ is a given point. If $\{\lambda_n\}$, $\{\theta_n\} \subset [0,1]$ satisfy appropriate conditions, then $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$ for each $l \in \{1, 2, \cdots, N\}$.

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1 Introduction and preliminaries

Throughout this paper we assume that E is a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) = \{f \in$

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$E^* : \langle x, f\rangle = \|x\|^2, \|x\| = \|f\|,\ x \in E\}$, where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex, then J is single-valved. In the sequel,we shall denote the single-valved duality mapping by j .

Let K be a subset of E and $T : K \to K$ a mapping. It is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in K$. It is called a *contraction* if there exists $\alpha \in (0, 1)$ such that $\|Tx - Ty\| \le \alpha\|x - y\|$ for all $x,\ y \in K$. It is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ with $k_n \ge 1$ and $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \le k_n\|x - y\|$ for all $n \ge 1$ and $x,\ y \in K$. It is called *asymptotically pseudocontractive* if there exists a sequence $\{k_n\} \subset [1, +\infty)$ such that $\lim_{n\to\infty} k_n = 1$, and there exists $j(x - y) \in J(x - y)$ such that the inequality

$$\langle T^n x - T^n y, j(x - y)\rangle \le k_n\|x - y\|^2,\ \ n \ge 1 \tag{1.1}$$

holds for all $x,\ y \in K$. It is easy to see that every asymptotically nonexpansive mapping is asymptotically pseudocontractive mapping.

Let $\{T_i\}_{i=1}^N : K \to K$ be a family of mappings. It is called *uniformly asymptotically regular* if for each $\varepsilon > 0$ there exists integer n_0, such that for all $n \ge n_0$ $\max_{1\le i,j\le N} \|T_i^{n+1}x - T_j^n x\| \le \varepsilon, \forall\ x \in K$. It is called *uniformly asymptotically regular with sequence* $\{\varepsilon_n\}$ if $\max_{1\le i,j\le N} \|T_i^{n+1}x - T_j^n x\| \le \varepsilon_n$, for all $x \in K$, where $\varepsilon_n \to 0$ as $n \to \infty$.

Mapping $T : K \to K$ is called *uniformly L-Lipschitzian* if there exists $L > 0$ such that $\|T^n x - T^n y\| \le L\|x - y\|,\quad n \ge 1,\ \ \forall x, y \in K$. It is called *pseudocontractive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \le \|x - y\|^2, \forall\ x, y \in K. \tag{1.2}$$

It follows from a result of Kato [4] that (1.2) is equivalent to

$$\|x - y\| \le \|x - y + r((I - T)x - (I - T)y)\| \tag{1.3}$$

for all $x, y \in K$ and all $r > 0$, where I denotes the identity mapping.

A mapping $T : K \to K$ is called *strongly pseudocontractive* if for each $x, y \in K$, there exists $j(x - y) \in J(x - y)$ and $k \in (0, 1)$ such that $\langle Tx - Ty, j(x - y)\rangle \le k\|x - y\|^2$.

Any sequence $\{x_n\}$ satisfying $\lim\limits_{n\to\infty} \|x_n - T_l x_n\| = 0$, $l = 1, \cdots, N$, is called an *approximate fixed point sequence* for a family mappings $\{T_i\}_{i=1}^N$.

The asymptotically pseudocontractive mapping has been studied by various authors(see e.g., [8,9] etc.). In 2003, Chidume and Zegeye[3] construct an approximate fixed point sequence for the class of asymptotically pseudocontractive mapping in Banach space,they proved the following theorem:

Theorem CZ[3] *Let K be a nonempty closed convex and bounded subset of a real Banach space E. Let $T : K \to K$ be uniformly L-Lipschitzian, uniformly* *asymptotically regular with sequence* $\{\varepsilon_n\}$ *and asymptotically pseudocontractive with sequence* $\{\kappa_n\}$ *such that for* $\lambda_n,\ \theta_n \in (0,1), \forall n \geq 1$, *the following conditions are satisfied:*

$(i) \lambda_n(1+\theta_n) \leq 1, \sum_{n=1}^{\infty} \lambda_n\theta_n = \infty, |\kappa_{n-1}-\kappa_n| = o(\lambda_n\theta_n^2),\ \ \lambda_n - 1 = o(\theta_n);$

$(ii) \lim_{n\to\infty} \theta_n = 0, \lim_{n\to\infty} \frac{\lambda_n}{\theta_n} = 0, \lim_{n\to\infty} \frac{|\frac{\theta_{n-1}}{\theta_n}-1|}{\lambda_n\theta_n} = 0, \lim_{n\to\infty} \frac{\varepsilon_{n-1}}{\lambda_n\theta_n^2} = 0.$

Let $\{x_n\}$ *be generated from* $x_1 \in K$ *by*

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n T^n x_n - \lambda_n\theta_n(x_n - x_1),\ \forall n \geq 1,$$

Then $\|x_n - Tx_n\| = 0$ *as* $n \to \infty$.

The above iterative sequences is generalized by WZ[11] as following:

$$x_{n+1} := \lambda_n\theta_n f(x_n) + [1-\lambda_n(1+\theta_n)]x_n + \lambda_n T^n x_n,\ x_1 \in K.$$

It is called viscosity iterative scheme. Recently, many authors studied this class of the iterative scheme because it is able to approximate solution of some variational inequalities (for example,1,2,10,11). By the sequence, WZ[11] prove the following result.

Theorem WZ [11] *Let* K *be a nonempty closed convex and bounded subset of a real Banach space* E, $f : K \to K$ *be a contraction.* $T : K \to K$ *is uniformly L-Lipschitzian, uniformly asymptotically regular with sequence* $\{\varepsilon_n\}$ *and asymptotically pseudocontractive with sequence* $\{\kappa_n\}$ *such that for* $\lambda_n,\ \theta_n \in (0,1)$, *the following conditions are satisfied:*

$(i) \lambda_n(1+\theta_n) \leq 1, \sum_{n=1}^{\infty} \lambda_n\theta_n = \infty,\ \ |\kappa_{n-1}-\kappa_n| = o(\lambda_n\theta_n^2),\ \lambda_n - 1 = o(\theta_n);$

$(ii) \lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \frac{\lambda_n}{\theta_n} = \lim_{n\to\infty} \frac{|\frac{\theta_{n-1}}{\theta_n}-1|}{\lambda_n\theta_n} = \lim_{n\to\infty} \frac{\varepsilon_{n-1}}{\lambda_n\theta_n^2} = 0.$

Let $\{x_n\}$ *be generated from* $x_1 \in K$ *by*

$$x_{n+1} := \lambda_n\theta_n f(x_n) + [1-\lambda_n(1+\theta_n)]x_n + \lambda_n T^n x_n$$

for all positive integer $n \geq 1$. *Then* $\|x_n - Tx_n\| = 0$, *as* $n \to \infty$.

The importance of approximate fixed point sequences is that once a sequence has been constructed and proved to be an appropriate fixed point sequence for a mapping T, then it is generally achieved that the sequence converges to a fixed point of T.

Question 1. Is it possible to construct an approximate fixed point sequence for a finite family of asymptotically pseudocontractive mappings in Banach spaces?

In this paper, we introduce the following iterative sequence $\{x_n\}$:

$$x_{n+1} := \lambda_n\theta_n f(x_n) + [1-\lambda_n(1+\theta_n)]x_n + \lambda_n T_n^n x_n,\ n \geq 1. \tag{1.4}$$

for a finite family of asymptotically pseudocontractive mappings $\{T_i\}_{i=1}^N : K \to K$, where $T_n = T_{n(modN)}$, $\{\lambda_n\}$, $\{\theta_n\} \subset [0,1]$, $x_1 \in K$ is a given point.

Our purpose in this paper is to give an affirmative answers to Question 1. Under suitable condition, we prove that $\|x_n - T_l x_n\| \to 0$ as $n \to \infty$ for $l = 1, 2, \cdots, N$.The results obtained can be regarded as extension of Theorem CZ and Theorem WZ.

In the sequel, we shall make use of the following Lemmas:

Lemma 1.1([6]). *Let E be a real normed linear space and J the normalized duality mapping on E. Then for each $x, y \in E$ and $j(x+y) \in J(x+y)$, we have*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$$

Lemma 1.2([5]). *Let $\{\rho_n\}$, $\{\alpha_n\}$, $\{\sigma_n\}$ be three sequences of nonnegative numbers satisfying conditions $\alpha_n \to 0$, $\frac{\sigma_n}{\alpha_n} \to 0$, as $n \to \infty$, $\Sigma_{n=1}^{\infty} \alpha_n = \infty$,*

$$\rho_{n+1}^2 \leq \rho_n^2 - \alpha_n \psi(\rho_{n+1}) + \sigma_n, \ \ n \geq 1$$

be given where $\psi : [0, +\infty) \to [0, +\infty)$ is a strictly increasing function such that it is positive on $(0, +\infty)$ and $\psi(0) = 0$. Then $\rho_n \to 0$ as $n \to \infty$.

2 Main results

Lemma 2.1. *Let K be a nonempty closed convex and bounded subset of a real Banach space E. Let $f : K \to K$ be a contraction mapping and $T_i : K \to K$ be N uniformly asymptotically regular, uniformly L-Lipschitzian and asymptotically pseudocontractive mapping with sequence $\{\kappa_n^{(i)}\}$, $i = 1, \cdots, N$. Suppose $\theta_n \in (0, 1)$ such that $\theta_n \to 0$ and $\kappa_n - 1 = o(\theta_n)$ as $n \to \infty$, where $\kappa_n = \max\{\kappa_n^{(1)}, \cdots, \kappa_n^{(N)}\}$. Then there exists a positive integer n_0 and a sequence $\{y_n\} \subset K$ such that when $n \geq n_0$, $\{y_n\}$ satisfies the following condition:*

$$y_n = A_n T_n^n y_n + (1 - A_n) f(y_n) \tag{2.1}$$

where $T_n^n = T_{n(mod\ N)}^n$, $A_n = \frac{1}{\kappa_n(1+\theta_n)}$. Further, $\|y_n - T_n y_n\| \to 0$, as $n \to \infty$.

Proof. Since $f : K \to K$ is a contraction, then there exists $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$, $x, y \in K$. By $\theta_n \to 0$ and $\kappa_n - 1 = o(\theta_n)$, as $n \to \infty$, there exists n_0 such that $\frac{\kappa_n - 1}{\kappa_n \theta_n} < t_0 < \frac{1}{\alpha} - 1$, $n \geq n_0$.

For $n \geq 1$, define the mapping $S_n : K \to K$ by $S_n(y) := A_n T_n^n y + (1 - A_n) f(y)$. Then $S_n : K \to K$ is continuous and strongly pseudocontractive, when $n \geq n_0$. By theorem 5 in Reich [7], S_n has a unique fixed point(say) $y_n \in K$, for $n \geq n_0$. This means that $y_n = A_n T_n^n y_n + (1 - A_n) f(y_n)$ has a unique solution for $n \geq n_0$. Since K is bounded, then we have

$$\|y_n - T_n^n y_n\| = (1 - A_n)\|f(y_n) - T_n^n y_n\| \to 0 (n \to \infty). \tag{2.2}$$

Given that for $n \geq n_0$,

$$\begin{aligned}\|y_n - T_n y_n\| &= \|(1-A_n)(f(y_n) - T_n y_n) + A_n(T_n^n y_n - T_n y_n)\| \\ &\leq (1-A_n)\|f(y_n) - T_n y_n\| + A_n(\|T_n^n y_n - T_n^{n+1} y_n\| + \|T_n^{n+1} y_n - T_n y_n\|) \\ &\leq (1-A_n)\|f(y_n) - T_n y_n\| + A_n(\|T_n^n y_n - T_n^{n+1} y_n\| + L\|T_n^n y_n - y_n\|). \qquad (2.3)\end{aligned}$$

By the uniformly asymptotic regularity of $\{T_i\}_{i=1}^N$ and (2.2-2.3) we have $\lim\limits_{n\to\infty} \|y_n - T_n y_n\| = 0$. Completing the proof of lemma 2.1. □

Theorem 2.2 *Let K be a nonempty closed convex and bounded subset of a real Banach space E, $f : K \to K$ a contraction. $T_i : K \to K$ are N uniformly L-Lipschitzian, asymptotically pseudocontractive with sequence $\{\kappa_n^{(i)}\}$, $i = 1, 2, \cdots, N$, and a family uniformly asymptotically regular mappings with sequence $\{\varepsilon_n\}$. Let $\{x_n\}$ be defined by*

$$x_{n+1} := \lambda_n \theta_n f(x_n) + [1 - \lambda_n(1+\theta_n)]x_n + \lambda_n T_n^n x_n, \ n \geq 1 \qquad (2.4)$$

where $T_n = T_{n(modN)}$, $x_1 \in K$ is a given point, $\{\lambda_n\}$, $\{\theta_n\}$ be two real sequences in [0, 1] satisfying the following conditions:

(i) $\lambda_n(1+\theta_n) \leq 1, \ \sum_{n=1}^{\infty} \lambda_n \theta_n = \infty, \ \lim_{n\to\infty} \theta_n = 0;$

(ii) $\lim_{n\to\infty} \frac{\lambda_n}{\theta_n} = 0, \ \lim_{n\to\infty} \frac{|\frac{\theta_{n-1}}{\theta_n} - 1|}{\lambda_n \theta_n} = 0, \lim_{n\to\infty} \frac{\varepsilon_{n-1}}{\lambda_n \theta_n^2} = 0;$

(iii) $|\kappa_{n-1} - \kappa_n| = o(\lambda_n \theta_n^2), \ \kappa_n - 1 = o(\theta_n), \ \kappa_n = \max\{\kappa_n^{(1)}, \cdots, \kappa_n^{(N)}\}.$

Then $\|x_n - T_l x_n\| \to 0$ as $n \to \infty$, for $l = 1, 2, \cdots, N$.

Proof. Step 1. We prove that $\|x_n - y_n\| \to 0$ as $n \to \infty$. Since $f : K \to K$ is a contraction, then there exists $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|, \ x, y \in K$. Since $\theta_n \to 0$ and $\kappa_n - 1 = o(\theta_n)$, as $n \to \infty$, there exists n_0 such that $\frac{\kappa_n - 1}{\kappa_n \theta_n} < t_0 < \frac{1}{\alpha} - 1$, $n \geq n_0$.

For $n \geq n_0$, let $\{y_n\}$ be the sequence defined as that in (2.1) and $B_n = \frac{1}{\kappa_n}$. Then from (2.4) and lemma 1.1 we get that, for $n \geq n_0$,

$$\begin{aligned}\|x_{n+1} - y_n\|^2 &= \|x_n - y_n + \lambda_n \theta_n f(x_n) - \lambda_n(1+\theta_n)x_n + \lambda_n T_n^n x_n\|^2 \\ &\leq \|x_n - y_n\|^2 + 2\lambda_n \langle \theta_n f(x_n) - (1+\theta_n)x_n + T_n^n x_n, j(x_{n+1} - y_n)\rangle \\ &= \|x_n - y_n\|^2 + 2\lambda_n \langle \theta_n(x_{n+1} - x_n - x_{n+1} + y_n + f(x_n) - y_n) - x_n + T_n^n x_n, j(x_{n+1} - y_n)\rangle \\ &\leq \|x_n - y_n\|^2 - 2\lambda_n \theta_n \|x_{n+1} - y_n\|^2 + 2\lambda_n \langle \theta_n(x_{n+1} - x_n) \\ &\quad + \theta_n(f(x_n) - y_n) - (x_n - T_n^n x_n) - (y_n - B_n T_n^n y_n) + (y_n - B_n T_n^n y_n) \\ &\quad - (x_{n+1} - B_n T_n^n x_{n+1}) + (x_{n+1} - B_n T_n^n x_{n+1}), j(x_{n+1} - y_n)\rangle \qquad (2.5)\end{aligned}$$

By the properties of y_n and the asymptotically pseudocontractivity of T_n, then

$$\theta_n(f(y_n) - y_n) - (y_n - B_n T_n^n y_n) + (1 - B_n) f(y_n) = 0, \qquad (2.6)$$

$$\langle (x_{n+1} - B_nT_n^n x_{n+1}) - (y_n - B_nT_n^n y_n), j(x_{n+1} - y_n)\rangle \geq 0, \tag{2.7}$$

for all $n \geq n_0$. It follows from (2.5) and (2.7) that, for all $n \geq n_0$,

$$\begin{aligned} &\|x_{n+1}-y_n\|^2 \\ &\leq \|x_n-y_n\|^2-2\lambda_n\theta_n\|x_{n+1}-y_n\|^2+2\lambda_n\langle\theta_n(x_{n+1}-x_n)+\theta_n(f(x_n)-y_n) \\ &\quad -(x_n-T_n^nx_n)-(y_n-B_nT_n^ny_n)+(x_{n+1}-B_nT_n^nx_{n+1}), j(x_{n+1}-y_n)\rangle \end{aligned} \tag{2.8}$$

Substituting (2.6) into (2.8) we have that

$$\begin{aligned} &\|x_{n+1}-y_n\|^2 \\ &\leq \|x_n-y_n\|^2-2\lambda_n\theta_n\|x_{n+1}-y_n\|^2+2\lambda_n\langle\theta_n(x_{n+1}-x_n)+\theta_n(f(x_n)-y_n)-x_n \\ &\quad +T_n^nx_n-\theta_n(f(y_n)-y_n)-(1-B_n)f(y_n)+x_{n+1}-B_nT_n^nx_{n+1}, j(x_{n+1}-y_n)\rangle \\ &= \|x_n-y_n\|^2-2\lambda_n\theta_n\|x_{n+1}-y_n\|^2+2\lambda_n\langle(\theta_n+1)(x_{n+1}-x_n) \\ &\quad +\theta_n(f(x_n)-f(x_{n+1})+f(x_{n+1})-f(y_n))+T_n^nx_n-B_nT_n^nx_n \\ &\quad +B_nT_n^nx_n-B_nT_n^nx_{n+1}-(1-B_n)f(y_n), j(x_{n+1}-y_n)\rangle \\ &\leq \|x_n-y_n\|^2-2\lambda_n\theta_n(1-\alpha)\|x_{n+1}-y_n\|^2+6\lambda_n\|x_{n+1}-x_n\|\|x_{n+1}-y_n\| \\ &\quad +2\lambda_n[(1-B_n)(\|T_n^nx_n\|+\|f(y_n)\|)+B_n\|T_n^nx_n-T_n^nx_{n+1}\|]\|x_{n+1}-y_n\| \\ &\leq \|x_n-y_n\|^2-2\lambda_n\theta_n(1-\alpha)\|x_{n+1}-y_n\|^2+2\lambda_n(3+L)\|x_{n+1}-x_n\|\|x_{n+1}-y_n\| \\ &\quad +2\lambda_n[(1-B_n)(\|T_n^nx_n\|+\|f(y_n)\|)]\|x_{n+1}-y_n\| \end{aligned} \tag{2.9}$$

Notice the fact that

$$\|x_{n+1}-x_n\| = \lambda_n\|\theta_nf(x_n)-(1+\theta_n)x_n+T_n^nx_n\| \triangleq \lambda_n\|v_n\|.$$

Since K is bounded, which implies that $\{x_n\}$,$\{f(x_n)\}$ and $\{T_n^nx_n\}$ are all bounded, then there exists $M_1 > 0$ such that

$$\max\{\|x_{n+1}-y_n\|, \|v_n\|, \|T_n^nx_n\|+\|f(y_n)\|\} \leq M_1,\ n \geq 1$$

Let $C_n = 2\lambda_n^2(3+L)M_1^2 + 2\lambda_n(1-B_n)M_1^2$. Hence, from (2.9) we get

$$\|x_{n+1}-y_n\|^2 \leq \|x_n-y_n\|^2-2\lambda_n\theta_n(1-\alpha)\|x_{n+1}-y_n\|^2+C_n. \tag{2.10}$$

Letting $\overline{T} := \frac{1}{\kappa_n}T_n^n$, then we have

$$\langle\overline{T}x-\overline{T}y, j(x-y)\rangle = B_n\langle T_n^nx-T_n^ny, j(x-y)\rangle \leq \|x-y\|^2,$$

therefore, $\overline{T}$ is a pseudocontractive mapping. Hence from (1.3) we have for $n \geq n_0$,

$$\|y_{n-1}-y_n\| \leq \|y_{n-1}-y_n+\frac{1}{\theta_n}\{(y_{n-1}-B_nT_n^ny_{n-1})-(y_n-B_nT_n^ny_n)\}\|. \tag{2.11}$$

By $B_n = 1/\kappa_n$ and (2.6) and (2.11) we get that

$$
\begin{aligned}
&\|y_{n-1}-y_n\| \\
&\le \|y_{n-1}-y_n+\frac{1}{\theta_n}\{y_{n-1}-B_{n-1}T_{n-1}^{n-1}y_{n-1}+B_{n-1}T_{n-1}^{n-1}y_{n-1}-B_nT_n^ny_{n-1}-y_n+B_nT_n^ny_n\}\| \\
&=\left\|\left(1-\frac{\theta_{n-1}}{\theta_n}\right)(y_{n-1}-f(y_{n-1}))+f(y_{n-1})-f(y_n)+\frac{1}{\theta_n}B_{n-1}(T_{n-1}^{n-1}y_{n-1}-T_n^ny_{n-1})\right. \\
&\qquad\left.+\frac{1}{\theta_n}(B_{n-1}-B_n)(T_n^ny_{n-1}-f(y_n))+\frac{1}{\theta_n}(1-B_{n-1})[f(y_{n-1})-f(y_n)]\right\| \\
&\le \left|1-\frac{\theta_{n-1}}{\theta_n}\right|\|y_{n-1}-f(y_{n-1})\|+\alpha\|y_{n-1}-y_n\|+\frac{\varepsilon_{n-1}B_{n-1}}{\theta_n} \\
&\qquad+\frac{1}{\theta_n}|B_{n-1}-B_n|\|T_n^ny_{n-1}-f(y_n)\|+\frac{\alpha}{\theta_n}(1-B_n)\|y_{n-1}-y_n\| \qquad (2.12)
\end{aligned}
$$

By $\frac{\alpha}{\theta_n}(1-B_n)=\frac{\alpha(\kappa_n-1)}{\theta_n\kappa_n}<\alpha t_0$, $n\ge n_0$, it follows from (2.12) that

$$
\begin{aligned}
\|y_{n-1}-y_n\| \le{}& \frac{1}{1-\alpha-\alpha t_0}\left|1-\frac{\theta_{n-1}}{\theta_n}\right|\|y_{n-1}-f(y_{n-1})\|+\frac{\varepsilon_{n-1}}{\theta_n\kappa_{n-1}(1-\alpha-\alpha t_0)} \\
&+\frac{1}{\theta_n(1-\alpha-\alpha t_0)}|B_{n-1}-B_n|\|T_n^ny_{n-1}-f(y_n)\| \qquad (2.13)
\end{aligned}
$$

Because K is bounded, which implies that $\{x_n\}$, $\{y_n\}$, $\{\|y_n-f(y_{n-1})\|\}$ and $\{\|T_n^ny_{n-1}-f(y_n)\|\}$ are all bounded. Thus, there exists $M_2>0$ such that

$$
\max\{2\|x_n-y_{n-1}\|+\|y_{n-1}-y_n\|,\frac{\|y_{n-1}-f(y_{n-1})\|}{1-\alpha-\alpha t_0},\frac{\|T_n^ny_{n-1}-f(y_n)\|}{1-\alpha-\alpha t_0}\}\le M_2.
$$

Let $M=\max\{M_1,M_2\}$, notice that

$$
\|x_n-y_n\|^2\le\|x_n-y_{n-1}\|^2+\|y_{n-1}-y_n\|M. \qquad (2.14)
$$

Hence, from (2.10), (2.13) and (2.14), we have that for $n\ge n_0$

$$
\begin{aligned}
\|x_{n+1}-y_n\|^2\le{}&\|x_n-y_{n-1}\|^2-2\lambda_n\theta_n(1-\alpha)\|x_{n+1}-y_n\|^2+2\lambda_n^2(3+L)M^2 \\
&+2\lambda_n(1-B_n)M^2+\frac{\varepsilon_{n-1}}{\theta_n\kappa_{n-1}}M^2+\left|1-\frac{\theta_{n-1}}{\theta_n}\right|M^2+\frac{1}{\theta_n}|B_{n-1}-B_n|M^2. \qquad (2.15)
\end{aligned}
$$

By Lemma 1.2 and the conditions on $\{\lambda_n\}$, $\{\theta_n\}$, we get that $\|x_{n+1}-y_n\|\to 0$ as $n\to\infty$. Consequently, $\|x_n-y_n\|\to 0$ as $n\to\infty$.

Step2. We prove that $\|x_n-T_nx_n\|\to 0$, as $n\to\infty$. Since

$$
\|x_n-T_nx_n\|\le(1+L)\|x_n-y_n\|+\|y_n-T_ny_n\|,
$$

from Lemma 2.1 and step 1, we know that $\|x_n-T_nx_n\|\to 0$ $(n\to\infty)$.

Step3. We prove that $\lim_{n\to\infty}\|x_n-T_lx_n\|=0$, $l=1,\cdots,N$. Since

$$
\|x_{n+1}-x_n\|=\lambda_n\|v_n\|\le\lambda_nM_1\to 0\ (n\to\infty).
$$

Thus, for each $j\in\{1,2,\cdots,N\}$, when $n\to\infty$, we have that

$$\|x_{n+j} - x_n\| \to 0,\ \|x_n - T_{n+j}x_n\| \to 0, \tag{2.16}$$

which implies that the sequence

$$\bigcup_{j=1}^{N} \{||x_n - T_{n+j}x_n||\}_{n=1}^{\infty} \to 0\ (n \to \infty).$$

For each $l \in \{1, 2, \cdots, N\}$, observe that

$$\begin{aligned}\{\|x_n - T_l x_n\|\}_{n=1}^{\infty} &= \{\|x_n - T_{n+(l-n)}x_n\|\}_{n=1}^{\infty} \\ &= \{\|x_n - T_{n+l_n}x_n\|\}_{n=1}^{\infty} \subset \bigcup_{l=1}^{N} \{\|x_n - T_{n+l}x_n\|\}_{n=1}^{\infty},\end{aligned}$$

where $l-n = l_n(modN)$, $l_n \in \{1, \cdots, N\}$. Therefore, we have $\|x_n - T_l x_n\| \to 0$ as $n \to \infty$. Completing the proof of Theorem 2.2. □

Remark 2.1 When $N \equiv 1$, Theorem 2.2 reduces to Theorem 3.2 in [11].

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