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Strong Convergence of Averaging Iterations of Nonexpansive Nonself-Mappings in Banach Spaces

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Abstract. Let E be a real uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of E which is also a sunny nonexpansive retract of E with P a sunny nonexpansive retraction. Let $T : C \longrightarrow E$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. For any $x, x_0, y, y_0 \in C$, let $\{x_n\}$ and $\{y_n\}$ be generated by the algorithm $x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j x_n$ and $y_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n y + (1-\alpha_n)(TP)^j y_n)$ for all $n \geq 0$ respectively, where $\{\alpha_n\} \subseteq [0,1]$ is a sequence satisfying certain conditions. Then the strong convergence of both $\{x_n\}$ and $\{y_n\}$ to an element of fixed points of T is proved. The results presented in this paper generalize, extend and improve the corresponding results of Matsushita and Kuroiwa [Strong convergence of averaging iterations of nonexpansive nonself-mappings, J. Math. Anal. Appl. 294 (2004) 206-214] and many others .

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1. Introduction

Let C be a nonempty closed convex subset of a Hilbert space H and let x be an element of C. Let T be a nonexpansive mapping from C into itself such that the set $F(T)$ of fixed points of T is nonempty. For each t with $0 < t < 1$, let x_t be a unique point of C which satisfies

$$x_t = tx + (1-t)Tx_t.$$

Browder [1] showed that $\{x_t\}$ converges strongly as $t \longrightarrow 0$ to the element of $F(T)$ which is nearest to x in $F(T)$. This result was extended to a Banach space by Reich

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[8] and Takahashi and Ueda [13]. On the other hand, Wittmann [15] showed that each sequence $\{x_n\}$ defined by

$$x_0 \in C, x_{n+1} = \alpha_{n+1} x + (1 - \alpha_{n+1}) T x_n \text{ for } n = 0, 1, 2, ...,$$

converges strongly to the element of $F(T)$ which is nearest to x if $\{\alpha_n\}$ satisfies $0 \leq \alpha_n \leq 1, \alpha_n \longrightarrow 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Using an idea of Browder [1], Shimizu and Takahashi [11] studied the convergence of the following approximated sequence for an asymptotically nonexpansive mapping in the framework of a Hilbert space:

$$x_n = \alpha_n x + (1 - \alpha_n) \frac{1}{n} \sum_{j=1}^{n} T^j x_n \text{ for } n = 0, 1, 2, \ldots, \tag{1.1.1}$$

where $\{\alpha_n\}$ is a real sequence satisfying $0 < \alpha_n < 1$ and $\alpha_n \longrightarrow 0$ as $n \longrightarrow \infty$. Shimizu and Takahashi [10] also studied the convergence of another iteration process for a family of nonexpansive mappings in the framework of a Hilbert space. The iteration process is a mixed iteration process of Wittmann's [15] and Shimizu and Takahash's [11]. For simplicity, we state their result for a nonexpansive mapping T with $F(T)$ is nonempty. They show that each sequence $\{x_n\}$ defined by

$$x_0 \in C, \;\; x_{n+1} = \alpha_n x + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n \text{ for } n = 0, 1, 2, \ldots, \tag{1.1.2}$$

converges strongly to the element of $F(T)$ which is nearest to x in $F(T)$ if $\{\alpha_n\}$ satisfies $0 \leq \alpha_n \leq 1, \alpha_n \longrightarrow 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then Shioji and Takahashi [9] extended their result to that of a real uniformly convex Banach space whose norm is uniformly Gâteaux differentiable. Furthermore, Song and Chen [12] also extended Shimizu and Takahashi's result [10] to a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping. But this approximation method is not suitable for some nonexpansive nonself-mappings. In the framework of a real Hilbert space, Matsushita and Kuroiwa [5] studied the strong convergence of the two sequences generated by

$$x_1 = x \in C, \;\; x_{n+1} = \alpha_n x + (1 - \alpha_n) P T x_n \text{ for } n = 1, 2, \ldots, \tag{1.1.3}$$

and

$$y_1 = y \in C, \;\; y_{n+1} = P(\alpha_n y + (1 - \alpha_n) T y_n) \text{ for } n = 1, 2, \ldots, \tag{1.1.4}$$

where P is the metric projection from H onto C, and T is a nonexpansive nonself-mapping from C into H. Then they proved that $\{x_n\}$ and $\{y_n\}$ converge strongly to fixed points of T when $F(T)$ is nonempty. Recently, Matsushita and Kuroiwa [6] also studied the strong convergence of two type iteration processes which are mixed iteration processes of (1.1.2), (1.1.3) and (1.1.4) in a real Hilbert space as follows: for $x, y \in C$,

$$x_0 \in C, \;\; x_{n+1} = \alpha_n x + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} (PT)^j x_n \text{ for } n = 0, 1, 2, \ldots, \tag{1.1.5}$$

and

$$(1.1.6)\quad y_0 \in C,\ \ y_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n y + (1-\alpha_n)(TP)^j y_n) \text{ for } n = 0,1,2,\dots,$$

where $\{\alpha_n\}$ is a real sequence satisfying $0 \le \alpha_n \le 1$, P is the metric projection of H onto C, and T is a nonexpansive nonself-mapping of C into H. Using the nowhere normal outward condition on T and the appropriate assumptions imposed upon the parameters sequences $\{\alpha_n\}$, they first proved that $\{x_n\}$ generated by (1.1.5) converges strongly as $n \longrightarrow \infty$ to an element of fixed point of T when $F(T)$ is nonempty, further they proved that the sequence $\{y_n\}$ generated by (1.1.6) converges strongly as $n \longrightarrow \infty$ to an element of fixed point of T when $F(T)$ is nonempty.

It is our purpose in this paper to extend Matsushita and Kuroiwa's result [6] to the framework of a Banach space. More precisely, we show that Matsushita and Kuroiwa's result [6] holds in a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable.

2. Preliminaries

Throughout this paper, it is assumed that E is a real Banach space with norm $\|\cdot\|$ and let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\| = \|x\| = \|f\|\}, \forall x \in E,$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing and $\mathbb{N}$ denotes the set of all positive integers. In the sequel, we shall denote the single-valued duality mapping by j, and denote $F(T) = \{x \in C : Tx = x\}$. When $\{x_n\}$ is a sequence in E, then $x_n \longrightarrow x$ (respectively $x_n \rightharpoonup x, x_n \overset{*}{\rightharpoonup} x$) will denote strong (respectively weak, weak*) convergence of the sequence $\{x_n\}$ to x. In a Banach space E, the following result (*the Subdifferential Inequality*) is well known [14, Theorem 4.2.1]: $\forall x, y \in E, \forall j(x+y) \in J(x+y), \forall j(x) \in J(x)$,

$$(2.2.1)\qquad \|x\|^2 + 2\langle y, j(x)\rangle \le \|x+y\|^2 \le \|x\|^2 + \langle y, j(x+y)\rangle.$$

A mapping T with domain $D(T)$ and range $R(T)$ in E is called *nonexpansive* (respectively, *contractive*) if for any $x, y \in D(T)$,

$$\|Tx - Ty\| \le \|x - y\|$$

(respectively, $\|Tx - Ty\| \le \beta\|x - y\|$ for some $0 \le \beta < 1$). A Banach space E is said to *strictly convex* if

$$\|x\| = \|y\| = 1, x \ne y \text{ imply } \frac{\|x+y\|}{2} < 1.$$

A Banach space E is said to be *uniformly convex* if for all $\varepsilon \in (0,2]$ there exits $\delta_\varepsilon > 0$ such that

$$\|x\| = \|y\| = 1 \text{ with } \|x - y\| \ge \varepsilon \text{ implies } \frac{\|x+y\|}{2} < 1 - \delta_\varepsilon.$$

Recall that the norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*), if the limit

$$\lim_{t\longrightarrow 0}\frac{\|x+ty\|-\|x\|}{t} \tag{2.2.2}$$

exists for each x, y on the unit sphere $S(E)$ of E. Moreover, if for each y in $S(E)$ the limit defined by (2.2.2) is uniformly attained for x in $S(E)$, we say that the norm of E is *uniformly Gáteaux differentiable.* It is well known that if E is a Banach space with a uniformly Gáteaux differentiable norm, then the mapping $J : E \longrightarrow E$ is single-valued and norm to weak star uniformly continuous on bounded sets of E [14, Theorem 4.3.6]. If C is a nonempty convex subset of a strictly convex Banach space E and $T : C \longrightarrow C$ is a nonexpansive mapping, then fixed point set $F(T)$ of T is a closed convex subset of C [14, Theorem 4.5.3].

Let $C \subseteq E$ be a closed convex and P a mapping of E onto C. Then P is said be *sunny* if $P(Px + t(x - Px)) = Px$ for all $x \in E$ and $t \geq 0$. A mappig P of E into E is said to be a *retraction* if $P^2 = P$. If a mapping P is a retraction, then $Px = x$ for every $x \in R(P)$, where $R(P)$ is the range of P. A subset C of E is said to be a *sunny nonexpansive retract* of E if there exists a sunny nonexpansive retraction of E onto C and it is said to be a *nonexpansive retract* of E if there exists a nonexpansive retraction of E onto C. If $E = H$ is a Hilbert space, each the metric projection is a sunny nonexpansive retraction from H to any closed convex subset of H. For more details, see ([3, 14]).

Let E be a smooth Banach space and let C be a nonempty closed convex subset of E. Let P be a sunny nonexpansive retraction of E onto C. Then, P is unique (see [7, 14]). For a nonself-mapping T of C into E, Matsushita and Takahashi [7] introduced the following condition:

$$Tx \in S_x^c, \forall x \in C, \tag{2.2.3}$$

where $S_x = \{y \in E : y \neq x, Py = x\}$ and P is a sunny nonexpansive retraction of E onto C.

Remark 2.1. [7, Remark 2.1] If C is a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space E, then for any $x \in E$, there exists a unique point $x_0 \in C$ such that

$$\|x_0 - x\| = \min_{y\in C}\|y - x\|.$$

The mapping P of E onto C defined by $Px = x_0$ is called the *metric projection.* Using the metric projection P, Halpern and Bergman [4] studied the following condition:

$$Tx \in \{y \in E : y \neq x, Py = x\}^c$$

for all $x \in C$. Such a condition is called the *nowhere-normal outward condition.* Note that if E is a Hilbert space, then the condition (2.2.3) and the nowhere-normal outward condition are equivalent.

It what follows, we shall make use of the following lemmas which can be found in [7].

Lemma 2.2. [7, Lemma 3.1] *Let C be a closed convex subset of a smooth Banach space E and let T be a mapping of C into E. Suppose that C is a sunny nonexpansive retract of E. If T satisfies the condition (2.2.3), then $F(T) = F(PT)$, where P is a sunny nonexpansive retraction from E onto C.*

Lemma 2.3. [7, Lemma 3.3] *Let C be a closed convex subset of a strictly convex Banach space E and let T be a nonexpansive mapping of C into E. Suppose that C is a sunny nonexpansive retract of E. If $F(T) \neq \emptyset$ then T satisfies the condition (2.2.3).*

The following theorem was proved by Reich [8] and Takahashi and Ueda [13].

Theorem 2.4. [8, 13] *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of E, and T a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then $F(T)$ is a sunny nonexpansive retract of C. Moreover, let $y \in C$ and let z_t be a unique point of C which satisfies $z_t = ty + (1-t)Tz_t$ for all $t \in (0,1)$. Then $\{z_t\}$ converges strongly as $t \longrightarrow 0$ to a fixed point of T.*

On the other hand, shioji and Takahashi [9] studied the convergence of another approximated sequence for a nonexpansive mapping in the framework of a Banach space.

Theorem 2.5. [9, Theorem 2] *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and C a nonempty closed convex subset of E. Let $T : C \longrightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{\alpha_n\}$ is a sequence of real numbers satisfying $0 \leq \alpha_n \leq 1$, $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ defined by (1.1.2) converges strongly as $n \longrightarrow \infty$ to a fixed point of T.*

In the sequel, we also need the following theorem which can be found in Bruck [2].

Theorem 2.6. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and $T : C \longrightarrow C$ be nonexpansive. Suppose that $T_n x = \frac{1}{n}\sum_{j=0}^{n-1} T^j x$ for all $x \in C$. Then*

$$\lim_{n\longrightarrow\infty} \sup_{x\in C} \|T_n x - T(T_n x)\| = 0.$$

3. Main Results

In this section, we prove two strong convergence theorems for a nonexpansive nonself-mapping in a uniformly convex Banach space.

Theorem 3.1. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and C a nonempty closed convex subset of E. Suppose that C is sunny nonexpansive retract of E with P a sunny nonexpansive retraction. Let T be a nonexpansive nonself-mapping of C into E such that $F(T)$ is nonempty. If $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ defined by (1.1.5) converges strongly as $n \longrightarrow \infty$ to $q \in F(T)$.*

Proof. Applying the Theorem 2.5 with the nonexpansive mapping PT, we obtain that $\{x_n\}$ converges strongly as $n \longrightarrow \infty$ to $q \in F(PT)$. Since $F(T) \neq \emptyset$, using Lemma 2.2 and 2.3, we obtain $F(T) = F(PT)$. The proof is complete. □

If in Theorem 3.1, $E = H$ is a real Hilbert space, then the requirement that C is a sunny nonexpansive retract of E is not necessary. In fact, we have the following corollary

Corollary 3.2. [6, Theorem 1] *Let H be a real Hilbert space, C a closed convex subset of H, P the metric projection of H onto C, and T a nonexpansive nonself-mapping of C into H such that $F(T)$ is nonempty. If* $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$, *then $\{x_n\}$ defined by* (1.1.5) *converges strongly as $n \longrightarrow \infty$ to $q \in F(T)$.*

Theorem 3.3. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and C a nonempty closed convex subset of E. Suppose that C is a sunny nonexpansive retract of E with P a sunny nonexpansive retraction. Let T be a nonexpansive nonself-mapping of C into E such that $F(T)$ is nonempty. If* $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$, *then the sequence $\{y_n\}$defined by (1.1.6) converges strongly as $\longrightarrow \infty$ to $q \in F(T)$.*

Proof. We first show that $\{y_n\}$ is bounded. Let $y \in C, z \in F(T)$ and $M = \max\{\|y-z\|, \|y_0 - z\|\}$. Then we have

$$\|y_1 - z\| = \|P(\alpha_0 y + (1-\alpha_0)y_0) - z\| \leq \alpha_0\|y - z\| + (1-\alpha_0)\|y_0 - z\| \leq M.$$

Thus, one easily shows by induction that $\|y_n - z\| \leq M$, for all integers $n \geq 0$, and hence $\{y_n\}$ is bounded. Suppose that $T_n := \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j$ for all $n \in \mathbb{N}$. Using the boundedness of $\{y_n\}$, we obtain that

$$\begin{aligned}
\|y_{n+1} - T_n y_n\| &= \|\frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n y + (1-\alpha_n)(TP)^j y_n) - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\| \\
&\leq \frac{1}{n+1}\sum_{j=0}^{n} \|P(\alpha_n y + (1-\alpha_n)(TP)^j y_n) - (PT)^j y_n\| \\
&\leq \frac{1}{n+1}\sum_{j=0}^{n} \|\alpha_n y + (1-\alpha_n)(TP)^j y_n - (TP)^j y_n\| \\
&= \alpha_n \frac{1}{n+1}\sum_{j=0}^{n} \|y - (PT)^j y_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \qquad (3.3.1)
\end{aligned}$$

Next we shall show that $\lim_{n\longrightarrow\infty} \|y_n - PTy_n\| = 0$. Take $w \in F(T)$ and define a subset D of C by $D = \{x \in C : \|x - w\| \leq r\}$ where $r = \max\{\|y - w\|, \|y_0 - w\|\}$. It is easily to see that D is a nonempty closed bounded convex subset of C, $PT(D) \subset D$ and $\{y_n\} \subset D$. Applying Theorem 2.6 with the nonexpansive mapping PT, we obtain that

$$\lim_{n\longrightarrow\infty} \sup_{x\in D} \|T_n x - PT(T_n x)\| = 0.$$

Furthermore,

$$\lim_{n\longrightarrow\infty}\|T_ny_n - PT(T_ny_n)\| \le \lim_{n\longrightarrow\infty}\sup_{x\in D}\|T_nx - PT(T_nx)\| = 0.$$

It follows from (3.3.1) and the last inequality that

$$\begin{aligned}\|y_{n+1} - PTy_{n+1}\| &\le \|y_{n+1} - T_ny_n\| + \|T_ny_n - PT(T_ny_n)\| + \|PT(T_ny_n) - PTy_{n+1}\| \\ &\le 2\|y_{n+1} - T_ny_n\| + \|T_ny_n - PT(T_ny_n)\| \\ &\longrightarrow 0 \text{ as } n \longrightarrow \infty. \qquad (3.3.2)\end{aligned}$$

Let $\{z_t\}$ be the net defined by

$$z_t = ty + (1-t)PTz_t \text{ where } 0 < t < 1.$$

Since PT is a nonexpansive mapping of C into itself, using Theorem 2.4 and Lemma 2.2, 2.3, we have that $\{z_t\}$ converges strongly to $q \in F(PT) = F(T)$ as $t \longrightarrow 0$. Next we shall show that

$$\limsup_{n\longrightarrow\infty}\langle y - q, j(y_n - q)\rangle \le 0.$$

From $z_t - y_n = t(y - y_n) + (1-t)(PTz_t - y_n)$, we get that

$$\begin{aligned}\|z_t - y_n\|^2 &= (1-t)\langle PTz_t - y_n, j(z_t - y_n)\rangle + t\langle y - y_n, j(z_t - y_n)\rangle \\ &= (1-t)\left[\langle PTz_t - PTy_n, j(z_t - y_n)\rangle + \langle PTy_n - y_n, j(z_t - y_n)\rangle\right] \\ &\quad + t\langle y - z_t, j(z_t - y_n)\rangle + t\|z_t - y_n\|^2 \\ &\le \|y_n - y_t\|^2 + \|PTy_n - y_n\|\|z_t - y_n\| + t\langle y - z_t, j(z_t - y_n)\rangle,\end{aligned}$$

and hence

$$\langle y - z_t, j(y_n - z_t)\rangle \le \frac{\|PTy_n - y_n\|}{t}\|z_t - y_n\|. \qquad (3.3.3)$$

From (3.3.2) and the boundedness of $\{z_t\}$ and $\{y_n\}$ we get that

$$\limsup_{n\longrightarrow\infty}\langle y - z_t, j(y_n - z_t)\rangle \le 0. \qquad (3.3.4)$$

On the other hand, since $\{z_t\}$ converges strongly to q, as $t \longrightarrow 0$ and the set $\{z_t - y_n\}$ is bounded, together with the fact that the duality map J is single-valued and norm-weak* uniformly continuous on bounded sets of a real Banach space E with uniformly Gâteaux differentiable norm, we have that for each n,

$$\langle z_t - q, j(y_n - z_t)\rangle \le \|z_t - q\|\|y_n - z_t\| \longrightarrow 0 \text{ as } t \longrightarrow 0 \qquad (3.3.5)$$

and

$$\langle y - q, j(y_n - q) - j(y_n - z_t)\rangle \longrightarrow 0 \text{ as } t \longrightarrow 0. \qquad (3.3.6)$$

Using (3.3.5) and (3.3.6), we get that

$$\begin{aligned}&|\langle y - q, j(y_n - q)\rangle - \langle y - z_t, j(y_n - z_t)\rangle| \\ = \; &|\langle y - q, j(y_n - q)\rangle - \langle y - q, j(y_n - z_t)\rangle - \langle y - z_t, j(y_n - z_t)\rangle \\ &+ \langle y - q, j(y_n - z_t)\rangle|\end{aligned}$$

$$= \quad |\langle y-q, j(y_n-q)-j(y_n-z_t)\rangle + \langle z_t-q, j(y_n-z_t)\rangle| \longrightarrow 0 \text{ as } t\longrightarrow 0.$$

Therefore for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t \in (0,\delta)$ and all $n \geq 0$, we have that

$$|\langle y-q, j(y_n-q)\rangle - \langle y-z_t, j(y_n-z_t)\rangle| < \varepsilon.$$

Hence noting (3.3.4), we get that

$$\limsup_{n\longrightarrow\infty}\langle y-q, j(y_n-q)\rangle < \limsup_{n\longrightarrow\infty}[\langle y-z_t, j(y_n-z_t)\rangle + \varepsilon] \leq \varepsilon.$$

Since ε is arbitrary, we get that

$$\limsup_{n\longrightarrow\infty}\langle y-q, j(y_n-q)\rangle \leq 0. \tag{3.3.7}$$

Then for any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$\langle y-q, j(y_n-q)\rangle \leq \varepsilon \tag{3.3.8}$$

for all $n \geq m$. On the other hand, from

$$y_{n+1}-q+\alpha_n(q-y) = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n y+(1-\alpha_n)(TP)^j y_n) - P(\alpha_n y+(1-\alpha_n)q),$$

and the inequality (2.2.1) we have

$$\begin{aligned}
\|y_{n+1}-q\|^2 &= \|y_{n+1}-q+\alpha_n(q-y)-\alpha_n(q-y)\|^2\\
&\leq \|y_{n+1}-q+\alpha_n(q-y)\|^2 - 2\alpha_n\langle q-y, j(y_{n+1}-q)\rangle\\
&\leq \|\frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n y+(1-\alpha_n)(TP)^j y_n) - P(\alpha_n y+(1-\alpha_n)q)\|^2\\
&\quad -2\alpha_n\langle q-y, j(y_{n+1}-q)\rangle\\
&= \left\{\frac{1}{n+1}\sum_{j=0}^{n}\|P(\alpha_n y+(1-\alpha_n)(TP)^j y_n) - P(\alpha_n y+(1-\alpha_n)q)\|\right\}^2\\
&\quad -2\alpha_n\langle q-y, j(y_{n+1}-q)\rangle\\
&\leq \left\{(1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}\|(TP)^j y_n - q\|\right\}^2 - 2\alpha_n\langle q-y, j(y_{n+1}-q)\rangle\\
&\leq (1-\alpha_n)^2\|y_n-q\|^2 + 2\alpha_n\langle y-q, j(y_{n+1}-q)\rangle\\
&\leq (1-\alpha_n)\|y_n-q\|^2 + 2\alpha_n\varepsilon\\
&= 2\varepsilon(1-(1-\alpha_n)) + (1-\alpha_n)\|y_n-q\|^2\\
&\leq 2\varepsilon(1-(1-\alpha_n)) + (1-\alpha_n)(2\varepsilon(1-(1-\alpha_{n-1})) + (1-\alpha_{n-1})\|y_{n-1}-q\|^2\\
&= 2\varepsilon(1-(1-\alpha_n)(1-\alpha_{n-1})) + (1-\alpha_n)(1-\alpha_{n-1})\|y_{n-1}-q\|^2
\end{aligned}$$

for all $n \geq m$. By induction, we obtain that

$$\|y_{n+1}-q\|^2 \leq 2\varepsilon(1-\prod_{k=m}^{n}(1-\alpha_k)) + \prod_{k=m}^{n}(1-\alpha_k)\|y_m-q\|^2.$$

Therefore, from $\sum_{n=0}^{\infty} \alpha_n = \infty$, we get that

$$\limsup_{n \longrightarrow \infty} \|y_{n+1} - q\| \leq 2\varepsilon.$$

By arbitrarity of ε, we conclude that $\{y_n\}$ converges strongly as $n \longrightarrow \infty$ to $q \in F(T)$. This completes the proof. $\square$

If in Theorem 3.3, $E = H$ is a real Hilbert space, then the requirement that C is a sunny nonexpansive retract of E can be omitted. In fact, we have the following corollary.

Corollary 3.4. [6, Theorem 2] *Let H be a real Hilbert space, C a closed convex subset of H, P the metric projection of H onto C. Let T be a nonexpansive nonself-mapping of C into H such that $F(T)$ is nonempty. If $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then $\{y_n\}$ defined by* (1.1.6) *converges strongly as $n \longrightarrow \infty$ to $q \in F(T)$.*

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