A General Note on $\delta$-Quasi Monotone
and Increasing Sequence

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Abstract

In the present paper, a general theorem concerning $\varphi - |C, \alpha, \rho, \gamma|_k$
summability factors of infinite series, has been proved.

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1 Introduction

A sequence of $(b_n)$ of positive numbers is said to be $\delta$-quasi monotone, if
$b_n \to 0$, $b_n > 0$ ultimately and $\Delta b_n \geq -\delta_n$, where $(\delta_n)$ is a sequence of positive
numbers (see[3]). A positive sequence $(b_n)$ is said to be almost increasing if
there exists a positive increasing sequence $(c_n)$ and two positive constants $A$
and $B$ such that $Ac_n \leq b_n \leq Bc_n$ (see[1]). Let $(\varphi_n)$ be a sequence of complex
numbers and let $\sum a_n$ be a given infinite series with partial sums $(s_n)$. We
denote by $\sigma_n^\alpha$ and $t_n^\alpha$ the n-th Cesàro means of order $\alpha$, with $\alpha > -1$, of the
sequence $(s_n)$ and $(na_n)$, respectively, i.e.

$$\sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_n^{\alpha-1} s_{\nu}$$

(1)

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n} A_n^{\alpha-1} \nu a_{\nu},$$

(2)

where

$$A_n^\alpha = O(n^\alpha), \: \alpha > -1, \: A_0^\alpha = 1 \: \text{and} \: A_{-n}^\alpha = 0 \: \text{for} \: n > 0.$$  

(3)
The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$ and $\alpha > -1$ if (see[7])
\[ \sum_{n=1}^{\infty} n^{k-1} \left| \frac{1}{\alpha} \frac{\sigma^\alpha_n - \sigma^\alpha_{n-1}}{|t_n|^{\alpha}} \right|^k < \infty. \] (4)

But since $t_n^\alpha = n \left( \frac{1}{\alpha} \frac{\sigma^\alpha_n - \sigma^\alpha_{n-1}}{|t_n|^{\alpha}} \right)$ (see[9]) condition (4) can also be written as
\[ \sum_{n=1}^{\infty} \frac{1}{n} |t_n|^{\alpha} < \infty. \] (5)

The series $\sum a_n$ is said to be summable $|C, \rho, \gamma|_k$, $k \geq 1$, if (see[8])
\[ \sum_{n=1}^{\infty} n^{\gamma(\rho k + k-1) - k} |t_n|^{\alpha} < \infty, \] (6)

where $\rho \geq 0$ and $\gamma$ is a real number.

The series $\sum a_n$ is said to be summable $\varphi - |C, \alpha|_k$, $k \geq 1$ and $\alpha > -1$, if (see [2] and [10])
\[ \sum_{n=1}^{\infty} n^{-k} |\varphi_n t_n|^{\alpha} < \infty, \] (7)

and it is said to be summable $\varphi - |C, \alpha, \rho, \gamma|_k$, $k \geq 1$, $\rho \geq 0$, $\gamma \geq 1$ if
\[ \sum_{n=1}^{\infty} n^{\gamma(\rho k + k-1) - 2k+1} |\varphi_n t_n|^{\alpha} < \infty. \] (8)

In the special case when $\gamma = 1$ and $\rho = 0$, $\varphi - |C, \alpha|_k$ summability is same as $\varphi - |C, \alpha|_k$ summability and when $\varphi_n = n^{1-\frac{1}{k}}$ (resp. $\varphi_n = n^{\beta+1-\frac{1}{k}}$) $\varphi - |C, \alpha, \rho, \gamma|_k$ summability is same as $|C, \alpha, \rho, \gamma|_k$ summability.

2 Known Result

Mazhar [12] proved the following theorem for $|C, 1|_k$ summability factors of infinite series.

**Theorem 2.1** Let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers $(B_n)$ such that it is $\delta$-quasi monotone with $\sum n\delta_n \log n < \infty$, $\sum B_n \log n$ is convergent and $|\Delta \lambda_n| \leq |B_n|$ for all $n$. If
\[ \sum_{n=1}^{m} \frac{1}{n} |t_n|^{\alpha} = O(\log m) \text{ as } m \to \infty, \] (9)

Later on Bor and Leindler [4] generalized the above theorem under weaker conditions in the following form for $\varphi - |C, \alpha|_k$ summability
Theorem 2.2 Let \((X_n)\) be an almost increasing sequence such that \(|\Delta X_n| = O \left( \frac{X_n}{n} \right)\) and \(\lambda_n \to 0\) as \(n \to \infty\). Suppose that there exists a sequence of numbers \((B_n)\) such that it is \(\delta\)-quasi monotone with \(\sum nX_n\delta_n < \infty\), \(\sum B_nX_n\) is convergent and \(|\Delta \lambda_n| \leq |B_n|\) for all \(n\). If there exists an \(\varepsilon > 0\) such that the sequence \(\left( n^{\varepsilon-k} |\varphi_n|^k \right)\) is non-increasing and if the sequence \((\omega_n^\alpha)\), defined by (see [13])

\[
\omega_n^\alpha = \begin{cases} 
|t_n^\alpha|, & \alpha = 1 \\
\max_{1 \leq \nu \leq n} |t_n^\alpha|, & 0 < \alpha < 1
\end{cases}
\tag{10}
\]

satisfies the condition

\[
\sum_{n=1}^{m} n^{-k} (\omega_n^\alpha |\varphi_n|)^k = O (X_m) \quad \text{as} \quad m \to \infty,
\tag{11}
\]

then the series \(\sum a_n\lambda_n\) is summable \(\varphi - |C, \alpha|_k\), \(k \geq 1\), \(\frac{1}{k} \leq \alpha \leq 1\).

3 Main Result

The aim of this paper is to generalize Theorem 2.2 for \(\varphi - |C, \alpha, \rho, \gamma|_k\) summability in the following form.

Theorem 3.1 Let \((X_n)\) be an almost increasing sequence such that \(|\Delta X_n| = O \left( \frac{X_n}{n} \right)\) as \(\lambda_n \to 0\) as \(n \to \infty\). Suppose that there exists a sequence of numbers \((B_n)\) such that it is \(\delta\)-quasi monotone with \(\sum nX_n\delta_n < \infty\), \(\sum B_nX_n\) is convergent and \(|\Delta \lambda_n| \leq |B_n|\) for all \(n\). If there exists an \(\varepsilon > 0\) such that the sequence \(\left( n^{\varepsilon-k} |\varphi_n|^k \right)\) is non-increasing and if the sequence \((\omega_n^\alpha)\) defined by (see [13]) as (10) satisfies the condition

\[
\sum_{n=1}^{m} n^{\gamma(\rho k+k-1)-2k+1} (|\varphi_n| \omega_n^\alpha)^k = O (X_m) \quad \text{as} \quad m \to \infty,
\tag{12}
\]

then the series \(\sum a_n\lambda_n\) is summable \(\varphi - |C, \alpha, \rho, \gamma|_k\), \(k \geq 1\), \(\frac{1}{k} \leq \alpha \leq 1\).

It is also a generalization of Bor and Özarslan [5] We need the following lemmas for the proof of our theorem.

Lemma 3.2 [6] If \(0 < \alpha \leq 1\) and \(1 \leq \nu \leq n\), then

\[
\left| \sum_{p=0}^{\nu} A_{n-p}^{\alpha-1} \right| \leq \max_{1 \leq m \leq \nu} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} \right|.
\tag{13}
\]

Lemma 3.3 Under the conditions regarding \((\lambda_n)\) and \((X_n)\) of the Theorem, we have

\[
|\lambda_n| X_n = O (1) \quad \text{as} \quad n \to \infty,
\tag{14}
\]
The statements proof of Lemma 3.3 are proved by Bor and Leindler [4] and hence is omitted.

**Lemma 3.4** Under the conditions pertaining to \((X_n)\) and \((B_n)\) of the Theorem, we have

\[
   nB_n X_n = O(1) \tag{15}
\]

\[
   \sum_{n=1}^{\infty} nX_n |\Delta B_n| < \infty. \tag{16}
\]

The statements proof of Lemma 3.4 are proved in Theorem 1 and Theorem 2 of Leindler [11] and hence is omitted.

**Proof of Theorem 3.1** Let \(T_n^\alpha\) be the n-th \((C, \alpha)\), mean of the sequence \((na_n\lambda_n)\). Then, by (2), we have

\[
   T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n} A_{n-p}^{\alpha-1} \nu a_p \lambda_n.
\]

Using Abel’s transformation, we get

\[
   T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n} \Delta \lambda_n \sum_{p=1}^{\nu} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \nu a_\nu,
\]

so that making use of Lemma 1, we have

\[
   T_n^\alpha \leq \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n-1} |\Delta \lambda_n| \left| \sum_{p=1}^{\nu} A_{n-p}^{\alpha-1} p a_p \right| + \frac{\lambda_n}{A_n^\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \nu a_\nu
\]

\[
   \leq \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n-1} A_n^{\alpha} \omega_\nu^{\alpha} |\Delta \lambda_n| + |\lambda_n| \omega_n^\alpha
\]

\[
   = T_{n,1}^\alpha + T_{n,2}^\alpha, \text{ say.}
\]

Since \(|T_{n,1}^\alpha + T_{n,2}^\alpha|^k \leq 2^k \left( |T_{n,1}^\alpha|^k + |T_{n,2}^\alpha|^k \right)\), to complete the proof of the theorem, it is sufficient to show that

\[
   \sum_{n=1}^{\infty} n^{\gamma(p\rho k-1)-2k+1} |\varphi_n T_{n,r}^\alpha|^k < \infty, \quad \text{for} \quad r = 1, 2 \quad \text{by (8)}.
\]

Now, when \(k > 1\), applying Hölder’s inequality with indices \(k\) and \(k'\), where \(\frac{1}{k} + \frac{1}{k'} = 1\), we get that

\[
   \sum_{n=1}^{m+1} n^{\gamma(p\rho k-1)-2k+1} |\varphi_n T_{n,1}^\alpha|^k
\]

\[
   \leq \sum_{n=2}^{m+1} n^{\gamma(p\rho k-1)-2k+1} (A_n^\alpha)^{-k} |\varphi_n|^k \left( \sum_{\nu=1}^{n-1} A_{\nu}^{\alpha} \omega_\nu^{\alpha} |\Delta \lambda_\nu| \right)^k
\]
\[\delta\text{-Quasi monotone and increasing sequence}\]

\[
\begin{align*}
&= O(1) \sum_{n=2}^{m+1} n^{\gamma(\rho k+1)-2k+1} n^{-\alpha k} |\varphi_n|^k \left\{ \sum_{\nu=1}^{n-1} \nu^{\alpha k} (\omega_\nu^\alpha)^k |B_\nu| \right\} \\
&\quad \times \left\{ \sum_{\nu=1}^{m+1} |B_\nu| \right\}^{k-1} \\
&= O(1) \sum_{\nu=1}^{m} \nu^{\alpha k} (\omega_\nu^\alpha)^k |B_\nu| \sum_{n=\nu+1}^{m+1} \frac{|\varphi_n|^k}{n^{\alpha k}} n^{\gamma(\rho k+1)-2k+1} \\
&= O(1) \sum_{\nu=1}^{m} \nu^{\alpha k} (\omega_\nu^\alpha)^k |B_\nu| \sum_{n=\nu+1}^{m+1} \frac{n^{\varepsilon+\gamma(\rho k+1)-2k+1}}{n^{\alpha k+\varepsilon}} |\varphi_n|^k \\
&= O(1) \sum_{\nu=1}^{m} \nu^{\alpha k} (\omega_\nu^\alpha)^k |B_\nu| \nu^{\varepsilon+\gamma(\rho k+1)-2k+1} |\varphi_\nu|^k \int_\nu^\infty \frac{dx}{x^{\alpha k+\varepsilon}} \\
&= O(1) \sum_{\nu=1}^{m} \nu |B_\nu| \nu^{\gamma(\rho k+1)-2k+1} (\omega_\nu^\alpha |\varphi_\nu|)^k \\
&\quad + O(1) m |B_m| \sum_{\nu=1}^{m} \nu^{\gamma(\rho k+1)-2k+1} (\omega_\nu^\alpha |\varphi_\nu|)^k \\
&= O(1) \sum_{\nu=1}^{m-1} |(\nu + 1) |\Delta B_\nu| - |B_\nu|| X_\nu + O(1) m |B_m| X_m \\
&= O(1) \sum_{\nu=1}^{m-1} \nu |\Delta B_\nu| X_\nu + O(1) m \sum_{\nu=1}^{m-1} |B_\nu| X_\nu + O(1) m |B_m| X_m \\
&= O(1) \sum_{\nu=1}^{m-1} \nu |\Delta B_\nu| X_\nu + O(1) \sum_{\nu=1}^{m-1} |B_\nu| X_{\nu+1} + O(1) m |B_m| X_m \\
&= O(1) \text{ as } m \to \infty
\end{align*}
\]

by the virtue of the hypotheses of the Theorem 3.1 and Lemma 3.4.
Again, since \(|\lambda_n| = O \left( \frac{1}{X_n} \right) = O(1)\), by (14) we have

\[
\begin{align*}
\sum_{n=1}^{m+1} n^{\gamma(\rho k+1)-2k+1} |\varphi_n T_n^\alpha|^{k} \\
&= \sum_{n=1}^{m} |\lambda_n|^{k-1} |\lambda_n| n^{\gamma(\rho k+1)-2k+1} (\omega_n^\alpha |\varphi_n|)^k \\
&= O(1) \sum_{n=1}^{m} |\lambda_n| n^{\gamma(\rho k+1)-2k+1} (\omega_n^\alpha |\varphi_n|)^k
\end{align*}
\]
\[
\begin{align*}
&= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{\nu=1}^{n} \nu^\gamma(p^k+k-1)^{-2k+1} (\omega^\alpha_{\nu} |\varphi_{\nu}|)^k \\
&\quad + O(1) \sum_{n=1}^{m} n^\gamma(p^k+k-1)^{-2k+1} (\omega^\alpha_{n} |\varphi_{n}|)^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| X_n + O(1) |\lambda_m| X_m \\
&= O(1) \sum_{n=1}^{m-1} |B_n| X_n + O(1) |\lambda_m| X_m \\
&= O(1) \text{ as } m \to \infty,
\end{align*}
\]

by the virtue of the hypotheses of Theorem 3.1 and Lemma 3.3.

Therefore, we get that
\[
\sum_{n=1}^{m} n^\gamma(p^k+k-1)^{-2k+1} |\varphi_{n} T^\alpha_{m,r}|^k = O(1), \text{ as } m \to \infty \text{ for } r = 1, 2.
\]

This completes the proof of the Theorem 3.1.

References


δ-Quasi monotone and increasing sequence


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