International Mathematical Forum, 3, 2008, no. 10, 457 - 462

Prime and Irreducible Ideals in Subtraction Algebras

Young Bae Jun

Department of Mathematics Education
Gyeongsang National University, Chinju 660-701, Korea
ybjun@nongae.gsnu.ac.kr

Kyung Ho Kim

Department of Mathematics
Chungju National University, Chungju 380-702, Korea
ghkim@cjnu.ac.kr

Abstract

The notion of prime and irreducible ideals in subtraction algebras is introduced. Characterizations of a prime ideal are given. Extension property of a prime ideal is established. Conditions for an ideal to be an irreducible ideal are given.

Mathematics Subject Classification: 03G25, 06B10

Keywords: Subtraction algebra, (prime, irreducible) ideal

1 Introduction

B. M. Schein [4] considered systems of the form $(\Phi; \circ, \setminus)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\setminus$" (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [5] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [2] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [3], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results.

In this paper, we investigate some properties of ideals in subtraction algebras. We give a condition for a nonempty subset to be an ideal. We introduce the notion of prime and irreducible ideals of a subtraction algebra, and we give a characterization of a prime ideal. We also provide a condition for an ideal to be a prime/irreducible ideal.

2 Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;

(S2) $x - (x - y) = y - (y - x)$;

(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [2]):

(p1) $(x - y) - y = x - y$.

(p2) $x - 0 = x$ and $0 - x = 0$.

(p3) $(x - y) - x = 0$.

(p4) $x - (x - y) \leq y$.

(p5) $(x - y) - (y - x) = x - y$.

(p6) $x - (x - (x - y)) = x - y$.

(p7) $(x - y) - (z - y) \leq x - z$.

(p8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.

(p9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.

(p10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.

3 Main Results.

Definition 3.1 (Jun et al. [2])**.** A nonempty subset A of a subtraction algebra X is called an *ideal* of X if it satisfies

(I1) $0 \in A$

(I2) $y \in A$ and $x - y \in A$ imply $x \in A$ for all $x, y \in X$.

Lemma 3.2. *Let A be an ideal of a subtraction algebra X. If $x \leq y$ and $y \in A$, then $x \in A$.*

Proof. It is straightforward. □

Lemma 3.3. *In a subtraction algebra X, the following inequality is valid.*

$$(x \wedge y) - (x \wedge z) \leq x \wedge (y - z).$$

Proof. For any $x, y, z \in X$, we have

$$\begin{aligned}(x \wedge y) - (x \wedge z) &= (x - (x - y)) - (x - (x - z)) \\ &\leq (x - z) - (x - y) \leq y - z. \qquad (3.1)\end{aligned}$$

On the other hand,

$$(x \wedge y) - (x \wedge z) \leq x \wedge y \leq x. \qquad (3.2)$$

Combining (3.1) and (3.2), we have $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$. □

Theorem 3.4. *Let A be an ideal of a subtraction algebra X. For any $w \in X$, the set*

$$A_w^{\wedge} := \{x \in X \mid w \wedge x \in A\}$$

is an ideal of X containing A.

Proof. Since $w \wedge 0 = w - (w - 0) = w - w = 0 \in A$, we have $0 \in A_w^{\wedge}$. Let $x, y \in X$ be such that $y \in A_w^{\wedge}$ and $x - y \in A_w^{\wedge}$. Then $w \wedge y \in A$ and $w \wedge (x - y) \in A$. Since $(w \wedge x) - (w \wedge y) \leq w \wedge (x - y)$ by Lemma 3.3, it follows from Lemma 3.2 and (I2) that $w \wedge x \in A$, that is, $x \in A_w^{\wedge}$. Hence $A_w^{\wedge}$ is an ideal of X. Now let $x \in A$. Since $w \wedge x \leq x$ by (p4), we have $w \wedge x \in A$ by Lemma 3.2. Therefore $x \in A_w^{\wedge}$, and so $A \subseteq A_w^{\wedge}$. This completes the proof. □

Theorem 3.5. *Let A be a nonempty subset of a subtraction algebra X such that*

(i) $x \in A$ and $y \leq x$ imply $y \in A$.

(ii) For $x, y \in A$, there exists $z \in A$ such that $x \leq z$ and $y \leq z$.

Then A is an ideal of X.

Proof. Since A is nonempty, we have $0 \in A$ by (i) and (p2). Let $x, y \in X$ be such that $y \in A$ and $x - y \in A$. Then, by (ii), there exists $z \in A$ such that $y \leq z$ and $x - y \leq z$. It follows from (p2) and [2, Lemma 3.10] that

$$x - z = (x - z) - 0 = (x - z) - (y - z) = (x - y) - z = 0$$

so that $x \leq z$. Since $z \in A$, it follows from (i) that $x \in A$. Hence A is an ideal of X. $\square$

Definition 3.6. Let X be a subtraction algebra. A *prime ideal* of X is defined to be an ideal P of X such that if $x \wedge y \in P$ then $x \in P$ or $y \in P$.

Theorem 3.7. *Let P be an ideal of a subtraction algebra X. Then the following are equivalent.*

(i) P is a prime ideal of X.

(ii) For any ideals A and B of X, $A \wedge B \subset P$ implies $A \subset P$ or $B \subset P$, where $A \wedge B := \{a \wedge b \mid a \in A, b \in B\}$.

Proof. Suppose that P is a prime ideal of X such that $A \wedge B \subset P$, where A and B are ideals of X. Assume that $A \not\subset P$ and $B \not\subset P$. Then there exist $x \in A \setminus P$ and $y \in B \setminus P$, and so $x \wedge y \in A \wedge B \subset P$. Since P is prime, it follows that $x \in P$ or $y \in P$, which is a contradiction. Consequently, $A \wedge B \subset P$ implies $A \subset P$ or $B \subset P$. Conversely assume that for any ideals A and B of X, $A \wedge B \subset P$ implies $A \subset P$ or $B \subset P$. Let $x, y \in X$ be such that $x \wedge y \in P$. Note from [2, Theorem 3.4] that $(x] := \{a \in X \mid a \leq x\}$ and $(y] := \{b \in X \mid b \leq y\}$ are ideals of X. Let $a \in (x]$ and $b \in (y]$. Then $a \leq x$ and $b \leq y$. It follows from $a \wedge b \leq a, b$ that $a \wedge b \leq x$ and $a \wedge b \leq y$ so that $a \wedge b \leq x \wedge y$. Since P is an ideal and $x \wedge y \in P$, by Lemma 3.2 we get $a \wedge b \in P$. Therefore $(x] \wedge (y] \subset P$, which implies that $(x] \subset P$ or $(y] \subset P$ by hypothesis. In particular, $x \in P$ or $y \in P$, and thus P is a prime ideal of X. $\square$

Theorem 3.8. *Every prime ideal of a subtraction algebra is a maximal ideal, that is, every prime ideal is not contained in any other proper ideal.*

Proof. Let P be a prime ideal of a subtraction algebra X. Suppose that P is not maximal. Then there exists a proper ideal A of X such that $P \subset A$ and $P \neq A$. Let $y \in X$ and consider $x \in A \setminus P$. Then

$$x \wedge (y - x) = x - (x - (y - x)) = x - x = 0 \in P,$$

and so $y - x \in P$ because P is a prime ideal and $x \notin P$. It follows from (I2) that $y \in A$, that is, $X = A$, which contradicts the assumption that A is proper. Hence P is a maximal ideal of X. $\square$

Proposition 3.9. *If A is a maximal ideal of a subtraction algebra X, then $x-y \in A$ or $y-x \in A$ for all $x, y \in X$.*

Proof. Let $x, y \in A$. Then $x-y \in A$ since A is an ideal. Assume that $x \in A$ and $y \in X \setminus A$. Since $x - y \leq x$, it follows from Lemma 3.3 that $x-y \in A$. Similarly, if $x \in X \setminus A$ and $y \in A$, then $y-x \in A$. Let $x, y \in X \setminus A$ and assume that $y - x \notin A$. Then the set

$$Q := \{z \in X \mid z - (y-x) \in A\}$$

is the least ideal of X containing A and $y-x$ (see [2, Theorem 3.11]). Since $y-x \notin A$, we have $A \neq Q$, and so $Q = X$ because A is maximal. Therefore $x-y \in Q$, that is, $(x-y)-(y-x) \in A$. Using (p2), (p3), (S3), and [2, Lemma 3.10], we get

$$\begin{aligned} x-y &= (x-y)-0 = (x-y)-((y-x)-y) \\ &= (x-(y-x))-y = (x-y)-(y-x) \in A. \end{aligned}$$

This completes the proof. □

Corollary 3.10. *If A is a prime ideal of a subtraction algebra X, then $x-y \in A$ or $y-x \in A$ for all $x, y \in X$.*

Now we consider the converse of Corollary 3.10.

Theorem 3.11. *Let A be an ideal of a subtraction algebra X such that $x-y \in A$ or $y-x \in A$ for all $x, y \in X$. Then A is a prime ideal of X.*

Proof. Let A be an ideal of X such that $x-y \in A$ or $y-x \in A$ for all $x, y \in X$. Assume that $x \wedge y \in A$. If $x-y \in A$, then $x-(x-y) = x \wedge y \in A$ and so $x \in A$ by (I2). If $y-x \in A$, then $y-(y-x) = x-(x-y) = x \wedge y \in A$. It follows from (I2) that $y \in A$. Hence A is a prime ideal of X. □

Hence we know that the notion of prime ideals and maximal ideals in a subtraction algebra coincide, and we restate it as a theorem.

Theorem 3.12. *Let A be an ideal of a subtraction algebra X. Then the following are equivalent.*

(i) *A is a prime ideal.*

(ii) *A is a maximal ideal.*

(iii) *$x-y \in A$ or $y-x \in A$ for all $x, y \in X$.*

Using Theorem 3.12, we can establish the extension property of prime ideals.

Corollary 3.13. *Let A and B be ideals of a subtraction algebra X such that $A \subset B$. If A is a prime ideal of X, then so is B.*

Definition 3.14. An ideal A of a subtraction algebra X is said to be *irreducible* if for any ideals C and D of X, $A = C \cap D$ implies $A = C$ or $A = D$.

Theorem 3.15. *Let A be an ideal of a subtraction algebra X and let $w \in X \setminus A$. Then there exists an irreducible ideal M of X such that $A \subset M$ and $w \notin M$.*

Proof. Let $\mathcal{A} := \{I \mid I \text{ is an ideal of } X, A \subset I, w \notin I\}$. Note that any chain of elements in $\mathcal{A}$ has an upper bound. Thus, by Zorn's Lemma, there exists a maximal element M in $\mathcal{A}$. Then $A \subset M$ and $w \notin M$. Let C and D be ideals of X such that $M = C \cap D$. Assume that $M \neq C$ and $M \neq D$. By the maximality of M, we have $w \in C$ and $w \in D$, that is, $w \in C \cap D$. Hence $M \neq C \cap D$, a contradiction. Therefore M is an irreducible ideal of X. $\square$

Theorem 3.16. *Let A be an ideal of a subtraction algebra X. Assume that for any $x, y \in X \setminus A$, there exists $z \in X \setminus A$ such that $z \leq x$ and $z \leq y$. Then A is an irreducible ideal of X.*

Proof. Suppose that A is not an irreducible ideal of X. Then there are two ideals C and D of X such that $A = C \cap D$, $A \neq C$, and $A \neq D$. Let $x \in C \setminus A$ and $y \in D \setminus A$. Using the assumption, there exists $z \in X \setminus A$ such that $z \leq x$ and $z \leq y$. Since $x \in C$ and $y \in D$, it follows from Lemma 3.2 that $z \in C \cap D = A$, which is a contradiction. Hence A is an irreducible ideal of X. $\square$

References

[1] J. C. Abbott, *Sets, Lattices and Boolean Algebras,* Allyn and Bacon, Boston 1969.

[2] Y. B. Jun, H. S. Kim and E. H. Roh, *Ideal theory of subtraction algebras,* Math. Bohemica (submitted)

[3] Y. B. Jun and H. S. Kim, *On ideals in subtraction algebras,* Math. Bohemica (submitted)

[4] B. M. Schein, *Difference Semigroups,* Comm. in Algebra **20** (1992), 2153–2169.

[5] B. Zelinka, *Subtraction Semigroups,* Math. Bohemica, **120** (1995), 445–447.

Received: October 9, 2007