International Mathematical Forum, 3, 2008, no. 5, 209 - 228

On the Surgery Assembly Map

Friedrich Hegenbarth

Dipartimento di Matematica, Università di Milano
Via Saldini n. 50, 20133 Milano, Italy
hegenbar@mat.unimi.it

Yuri V. Muranov

Division de Estudios de Postgrado, Universidad Tecnologica
de la Mixteca, km.2.5 Carretera a Acatlima, Huajuapan de Leon
Oaxaca, C.P. 69000 Mexico
and
The Abdus Salam International Centre
for Theoretical Physics, Trieste, Italy

Abstract

The question of whether an element of the Wall group is the surgery obstruction of a degree-one normal map of closed manifolds is one of the basic problems in surgery theory. In the present paper we obtain new forbidding invariants in this problem for the case of finite 2-groups with arbitrary orientation. Our approach is based on the conception of iterated Browder-Livesay invariants developed by Hambleton and Kharshiladze jointly with the results on closed manifold surgery problem in oriented case obtained by Hambleton, Milgram, Taylor, and Williams. We consider several examples and give an application of our results to the surgery on filtered manifolds.

Mathematics Subject Classification: 57R67, 19J25

Keywords: Surgery assembly map, splitting problem, Browder–Livesay invariants, normal maps, closed manifolds surgery problem, manifolds with filtration, Browder-Quinn surgery obstruction groups

1 Introduction

Throughout the paper we consider finitely presented groups π with given subgroup $\rho \subset \pi$ of index 2. As usual, $B\pi = K(\pi, 1)$ denotes the classifying

space of π. We suppose that the group π equipped with an orientation homomorphism $w : \pi \to \{\pm 1\}$ and $L_n(\pi)$ denotes the Wall surgery obstruction group $L^s_n(\pi, w)$. For a manifold X we suppose that the orientation map $w : \pi = \pi_1(X) \to \{\pm 1\}$ coincides with the Stiefel-Whitney character. Recall that the decorated groups $L^U_*(\pi)$ are defined for any involution invariant subgroup $U \subset K_1(Z[\pi])$ (see [6], [10], [15], [19], [22], and [23]) where Z denotes the ring of integers. Let S be the subgroup $K_1(Z[\pi])$ generated by elements $\pm g, g \in \pi$. Setting $K = K_1(Z[\pi]), X = SK_1(Z[\pi]) = \text{Ker } (K_1(Z[\pi]) \to K_1(Q[\pi]))$, and $Y = X + S$, we obtain isomorphisms [22]

$$L^s_{2n}(\pi) \cong L^S_{2n}(Z[\pi]), \qquad L'_{2n}(\pi) \cong L^Y_{2n}(Z[\pi]),$$

while, in odd dimensions, the right-hand sides would be factorized by the subgroup ($\cong Z_2$) generated by the class of the automorphism

$$\tau = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}.$$

We are going to study the problem of representing $x \in L_n(\pi)$ as a surgery obstruction $\sigma_*(f, b)$ of a degree-one normal map

$$(f, b) : (M^n, \nu_M) \to (X^n, \nu_X), \tag{1}$$

of closed topological manifolds, where $b : \nu_M \to \nu_X$ is a bundle map $f : M^n \to X^n$ of the stable normal bundle of M in Euclidean space to a bundle over X. In what follows we shall not mention the maps of stable bundles to avoid confusion.

It is necessary to note here, that any element $x \in L_n(\pi)$ is represented by a normal map of closed manifolds with boundaries. The results of [23] provide the following representation. Choose a closed $(n-1)$-manifold X^{n-1} with $\pi_1(X^{n-1}) = \pi$. Then there is a normal map

$$(F, B) : (W^n; \partial_0 W, \partial_1 W) \to (X \times I; X \times \{0\}, X \times \{1\})$$

with $\sigma^*(F, B) = x$ where $\partial_0 W = X$, $F|_{\partial_0 W} = \text{Id}$, $\partial_1 W = M^n$, and $f = F|_M : M \to X$ is a simple homotopy equivalence.

The Assembly map

$$A : H_n(B\pi; \mathbf{L}_\bullet) \to L_n(\pi) \tag{2}$$

fits in the algebraic surgery exact sequence of Ranicki [18] for the space $B\pi$

$$\cdots \to L_{n+1}(\pi) \to \mathcal{S}_{n+1}(B\pi) \to H_n(B\pi; \mathbf{L}_\bullet) \xrightarrow{A} L_n(\pi) \to \cdots \tag{3}$$

where $\mathcal{S}_{n+1}(B\pi)$ is the structure set of the topological space $B\pi$ and $\mathbf{L}_\bullet$ is an Ω-spectrum [18] (see also [19] and [23]). More precisely, the spectrum $\mathbf{L}_\bullet$ is

the 1-connected cover of the simply connected surgery Ω-spectrum $\mathbf{L}(1)$. Note that $\pi_n(\mathbf{L}(1)) = L_n(1)$ for $n > 0$ and $\mathbf{L}_{\bullet 0} \simeq G/TOP$.

Let $\Omega_n^{STOP}(X, w) = \Omega_n^{STOP}(X)$ be the oriented topological bordism group of a space X. The surgery obstructions define a natural in π homomorphism [23, Theorem 13B.3] (see also [18])

$$\Theta : \Omega_n^{STOP}(B\pi \times G/TOP, B\pi \times \{*\}) \longrightarrow L_n(\pi) \tag{4}$$

which factors through the assembly map A (2). It follows from this that the image $C_n(\pi) \subset L_n(\pi)$ of the Assembly map consists of the surgery obstructions $\sigma_*(f, b) \in L_n(\pi)$ where (f, b) is a normal map of closed manifolds (1) with a given orientation map w. For any involution invariant subgroup $S \subset U \subset K_1(\mathrm{Z}[\pi])$, we shall denote by $C_n^U(\pi) \subset L_n^U(\pi)$ the image of the group $C_n(\pi)$ for the natural map

$$L_n(\pi) = L_n^s(\pi) \to L_n^U(\pi).$$

Thus we have the Assembly map for decoration U

$$A^U : H_n(B\pi, \mathbf{L}_\bullet) \to L_n^U(\pi), \quad \mathrm{Im}\ A^U = C_n^U(\pi). \tag{5}$$

we shall denote by $\sigma_n^U(f, b)$ the image of surgery obstruction of the map (1) in the group $L_n^U(\pi)$. In particular, the obstruction $\sigma_n'(f, b) = 0$ if and only if the normal map (1) is normally cobordant to *a weakly simple homotopy equivalence* [10].

The problem of the computation of the Assembly map is one of the significant problems of geometrical topology. For infinite groups without torsion we have the Novikov Conjecture. For the case of finite groups the significant results in the oriented case were obtained in [10]. In particular, the following theorem gives the very explicit result for finite groups with trivial orientations [10].

Theorem 1.1 *Let (1) be a closed manifold surgery problem with π finite and n odd. Then (f, b) is normally cobordant to a weakly simple homotopy equivalence if and only if either*

i) $n \equiv 1(\mathrm{mod}4)$ and $\sigma_*(f, b)'$ maps to zero under transfer-projection to all quaternion subquotients $Q(2^k)$ of the group π, or

ii) $n \equiv 3(\mathrm{mod}\,4)$ and $\sigma_*(f, b)'$ maps to zero under transfer-projection to all $Z/2$ quotients of the group π.

In Theorem 1 a group π' is a subquotient of a group π if $\pi' = \rho\,/\rho'$ where $\rho \subset \pi$ is a subgroup and ρ' is a normal subgroup of the group ρ, and the group $Q(2^k)$ is the generalized quaternion group [10].

For the cases of nontrivial orientation and infinite groups with torsion there are only a few results in computing of the Assembly map. The approach is

based on the conception of Browder-Livesay invariants (see [2], [3], [5], [8], [13], and [14]) is applicable to any finitely generated group with a subgroup of index 2 and sometimes gives very good results [12], [14]. The computation of the Assembly map for elementary abelian 2-groups in [12] and [14] is based on the concept of three possible types element $x \in L_*(\pi)$ [12], [13], [14]. The elements of the first type are detected by iterated Browder-Livesay invariants and cannot be realized by normal maps of closed manifolds. All iterated Browder-Livesay invariants of the elements of the second type are trivial but these elements give an infinite nontrivial number of nontrivial obstructions going down along the system of one-dimensional submanifolds (see definition in section 2). The theorem about non realization of the elements of the second type was proved by Kharshiladze [13], [14]. We note that in all known cases the elements of the second type, in fact, come from an elementary abelian 2-group [15], [16].

In the present paper, joining methods of surgery on filtered manifolds together with Theorem 1, we prove several nonrealization theorems of general type. As an application, we obtain that the results of Theorem 1 provide the nonrealization of the elements of the second type of any elementary abelian 2-group with arbitrary orientation. Then we describe the relation of the given approach to Browder-Quinn surgery obstruction groups for filtered manifolds (see [7], [17], and [24]) and give several examples of applications of obtained results. In particular, we obtain several theorems about elements of Browder-Quinn surgery obstruction groups that cannot be realized by a normal map to a filtered manifold.

2 Preliminaries

The algebraic surgery exact sequence is defined for any space X equipped with an orientation map $w : \pi_1(X) \to \{\pm 1\}$. Algebraic surgery sequence is the homotopy long exact sequence of the cofibration

$$X_+ \wedge \mathbf{L}_\bullet \to \mathbf{L}(\pi_1(X)). \tag{6}$$

where $\mathbf{L}(\pi)$ is the four-periodic surgery Ω-spectrum of the group π constructed by Ranicki with $\pi_n(\mathbf{L}(\pi)) = L_n(\pi)$, $n \geq 0$ [18], [19].

Let X be a closed n-dimensional topological manifold. An orientation preserving simple homotopy equivalence $f\ M \to X$ is called a *homotopical triangulation* of the manifold X. Two homotopical triangulations $f_i : M_i \to X (i = 1, 2)$ are equivalent if there exists an homeomorphism $h : M_1 \to M_2$ fitting in the homotopy commutative diagram [18], [19], [23]

$$\begin{array}{ccc} M_1 & \stackrel{h}{\to} & M_2 \\ & {\scriptstyle f_1}\searrow & \downarrow{\scriptstyle f_2} \\ & & X \end{array} \tag{7}$$

The set of equivalence classes $\mathcal{S}^{TOP}_{n+1}(X)$ fits into the surgery exact sequence [18], [19], [23]

$$\cdots \to [\Sigma X, G/TOP] \to L_{n+1}(\pi) \to \mathcal{S}^{TOP}_{n+1}(X) \to [X, G/TOP] \to L_n(\pi). \quad (8)$$

For a closed n-dimensional topological manifold X the surgery exact sequence (8) is isomorphic to the left part (from the group $L_n(\pi)$) of the algebraic surgery exact sequence of the space X with $H_n(X; \mathbf{L}_\bullet) \cong [X, G/TOP]$ and $\mathcal{S}_i(X) = \mathcal{S}^{TOP}_i(X)$ [18], [19], [23].

Now let X^n be a closed topological n-manifold with a fundamental group π with the orientation homomorphism w and $n \geq 6$. Consider a map

$$\phi : X^n \to RP^N \quad (9)$$

to a real projective space of high dimension which induces an epimorphism of fundamental groups $\pi \to \mathrm{Z}/2$ that coincides with w and has a kernel ρ. Let $i : \rho \hookrightarrow \pi$ be the inclusion of this kernel. Without loss of generality we can suppose that the map ϕ is transversal to $RP^{N-1} \subset RP^N$. Let

$$Y^{n-1} = (\phi_n)^{-1}(RP^{N-1}) \subset X^n \quad (10)$$

be a transversal preimage of RP^{N-1} and U be a tubular neighborhood of Y in X with boundary ∂U. For $(n-1) \geq 5$ we can suppose that the induced map $\pi_1(Y) \to \pi_1(X)$ is an isomorphism [23, §11]. The obtained pair of manifolds (X, Y) is called *a Browder-Livesay pair* and the submanifold $Y \subset X$ is called *a characteristic submanifold with respect to the inclusion* i. Note that i defines the orientation homomorphism $w : \pi \to \{\pm 1\}$ and Y is the one-sided submanifold in X. The map (9) is called a *characteristic map of the inclusion* $i : \rho \to \pi$. Let

$$F = \begin{pmatrix} \pi_1(\partial U) & \to & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(U) & \to & \pi_1(X) \end{pmatrix} \quad (11)$$

be the push-out square of fundamental groups in which horizontal maps are isomorphisms and vertical maps are inclusions of index 2.

By definition, a simple homotopy equivalence $f : M \to X$ *splits along* Y if it is homotopic to a map $g : M \to X$ which is transversal to the submanifold Y with $N = g^{-1}(Y)$, and whose restrictions $g|_N$ and $g|_{(M \setminus N)}$ are simple homotopy equivalences [18, §7], [23, §11]. The splitting obstruction groups

$$LN_{n-1}(\pi_1(X \setminus Y) \to \pi_1(X)) = LN_{n-1}(\rho \to \pi)$$

for a Browder-Livesay manifold pair (X, Y) are defined [3], [5], [8], [18, §7], [23, §11]. The algebraic definition of the LN_*-groups is given in [20]. These

groups depend functorially on the oriented inclusion $\rho \to \pi$ and the dimension $n - q \bmod 4$. Note that in the square F (11) the upper horizontal map and vertical maps agree with orientations. The bottom horizontal maps preserve the orientations on the images of the vertical maps and reverse these orientations outside these images. We shall denote the group with orientation $\pi_1(X)$ by π^+ and the group $\pi_1(Y)$ by π^-. We shall omit "+" if this doesn't lead to misunderstanding.

An inclusion $i : \rho \to \pi$ of index 2 defines the induced map

$$i_* : L_*(\rho) \to L_*(\pi), \tag{12}$$

and the transfer map

$$i^! : L_*(\pi) \to L_*(\rho), \tag{13}$$

of surgery obstruction groups. The maps (12) and (13) are realized on the spectra level (see [2], [6], [9], [15], [18], and [19]) by the maps fitting in cofibrations of spectra

$$\mathbf{L}(\rho) \to \mathbf{L}(\pi) \to \mathbf{L}(i_*) \tag{14}$$

and

$$\mathbf{L}(\pi) \to \mathbf{L}(\rho) \to \mathbf{L}(i^!), \tag{15}$$

and we have isomorphisms

$$\pi_n(\mathbf{L}(i_*)) = L_n(i_*) \ \text{ and } \ \pi_n(\mathbf{L}(i^!)) = L_n(i^!).$$

For a Browder-Livesay pair (X, Y) we have the cofibration $\partial U \to Y$. The geometrically defined transfer map and the square F (11) provide a homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \mathbf{L}(\pi_1(Y)) \to & \Omega^1\mathbf{L}(\pi_1(\partial U) \to \pi_1(U)) & \to & \Omega^1\mathbf{L}(\pi_1(X \setminus Y) \to \pi_1(X)) \\ \searrow & \downarrow & & \downarrow \\ & \mathbf{L}(\pi_1(\partial U)) & \to & \mathbf{L}(\pi_1(X \setminus Y)), \end{array} \tag{16}$$

where the left maps are the transfer maps on the spectra level and the right square is induced from the square F [2], [8], [11], [18], [19], [23]. The spectrum $\mathbf{LN}(\rho \to \pi)$ for the Browder-Livesay groups fits in the cofibration

$$\Omega\mathbf{L}(\pi_1(Y)) \to \Omega^2\mathbf{L}(\pi_1(X \setminus Y) \to \pi_1(X)) \to \mathbf{LN}(\rho \to \pi) \tag{17}$$

where the left map is the loop map of the composition from (16).

For a manifold pair (X, Y) the obstruction group $LP_*(F)$ is defined to obtain a split simple homotopy equivalence in the class of a normal cobordism of a normal map (1.1) [19], [23]. The spectrum $\mathbf{LP}(F)$ for these groups fits the cofibration

$$\Omega\mathbf{L}(\pi_1(Y)) \to \Omega\mathbf{L}(\pi_1(X \setminus Y)) \to \mathbf{LP}(F) \tag{18}$$

where the left map is the looping map of the composition from (16). The homotopy commutativity of diagram (16) induces a map

$$\mathbf{LN}(\rho \to \pi) \to \mathbf{LP}(F)$$

of spectra fitting into the homotopy commutative diagram of spectra [2], [11], [19]

$$\begin{array}{ccccc} \Omega\mathbf{L}(\pi_1(Y)) & \to & \Omega^2\mathbf{L}(\pi_1(X \setminus Y) \to \pi_1(X)) & \to & \mathbf{LN}(\rho \to \pi) \\ \downarrow= & & \downarrow & & \downarrow \\ \Omega\mathbf{L}(\pi_1(Y)) & \to & \Omega\mathbf{L}(\pi_1(X \setminus Y)) & \to & \mathbf{LP}(F). \end{array} \tag{19}$$

The right square in diagram (19) is a pullback since the fibers of horizontal maps are naturally homotopy equivalent. The homotopy long exact sequences of the maps of the right square from diagram (19) provides a commutative braid of exact sequences

$$\begin{array}{ccccccccc} \to & & L_n(\rho) & \xrightarrow{i_*} & L_n(\pi) & \xrightarrow{\partial} & LN_{n-2}(\rho \to \pi) & \to \\ & \nearrow & & \searrow & s \nearrow & \searrow & & \nearrow & \searrow \\ & & & LP_{n-1}(F) & & \Gamma \downarrow & L_n(i_*) & & \\ & \searrow & & \nearrow & q \searrow & \nearrow & & \searrow & \nearrow \\ \to & & LN_{n-1}(\rho \to \pi) & \xrightarrow{c} & L_{n-1}(\pi^-) & \xrightarrow{i^!_-} & L_{n-1}(\rho) & \to \end{array} \tag{20}$$

where $LP_{n-1}(F) \cong L_n(i^!_-)$ [2], [8], [19], [20]. The upper and bottom rows of diagram (20) are chain complexes. The map Γ denotes the following isomorphism of the corresponding homology groups. Take an element $x \in L_n(\pi)$ with $\partial x = 0$. This element represents a homology class

$$[x] \in \text{Ker } \partial / \text{Im } i_*.$$

We represent the class $\Gamma([x])$ by an element $q(y)$ where $y \in LP_{n-1}(F)$ and $s(y) = x$. Note that the class $\Gamma([x])$ consists of the elements $\{qs^{-1}(z) | z \in [x]\}$. Diagram (20) has period 4 hence we can consider all subscripts in this diagram mod4.

Now we give a new definition of the elements of the second type in the group $L_n(\pi)$. Our definition is a natural generalization of the approach [7], [8], [13], [14], [17]. Let us consider the commutative diagram $\mathcal{D}_k$ of groups

$$\begin{array}{ccccccccccc} \cdots \searrow & & \swarrow & \overset{\rho_2}{} & \searrow & & \swarrow & \overset{\rho_1}{} & \searrow & & \swarrow & \overset{\rho_0}{} & \searrow \\ & \pi_3 & & \xrightarrow{\cong} & & \pi_2 & & \xrightarrow{\cong} & & \pi_1 & & \xrightarrow{\cong} & & \pi_0 \end{array} \tag{21}$$

where all the maps $\rho_i \to \pi_i$ and $\rho_i \to \pi_{i+1}$ are index 2 inclusions. The subscript k is the largest subscript of the groups π_* for finite diagram (21) and $k = \infty$ for

the infinite diagram. We shall suppose that all the groups from diagram (21) are equipped with orientation homomorphism in the following way that the horizontal maps preserve the orientations on the images of ρ_i and reverse the orientations outside these images. All the sloping maps preserve orientations. Denote by

$$F_j = \begin{pmatrix} \rho_j & \stackrel{=}{\to} & \rho_j \\ \downarrow & & \downarrow \\ \pi_{j+1} & \to & \pi_j \end{pmatrix}$$

the square that is defined by the triangle with the upper group ρ_j from diagram (21). The square F_j defines diagram (20) for the inclusion $\rho_j \to \pi_j$. Putting together central squares from these diagrams we obtain the following commutative diagram

$$\begin{array}{ccccc}
\xrightarrow{i_*} & & L_n(\pi_0) & & \xrightarrow{\partial_0} \\
 & s\nearrow & & \searrow & \\
LP_{n-1}(F_0) & & \Gamma\downarrow & & L_n(\rho_0 \to \pi_0) \\
 & q\searrow & & \nearrow & \\
\longrightarrow & & L_{n-1}(\pi_1) & & \xrightarrow{\partial_1} \\
 & s\nearrow & & \searrow & \\
LP_{n-2}(F_1) & & \Gamma\downarrow & & L_{n-1}(\rho_1 \to \pi_1) \\
 & q\searrow & & \nearrow & \\
\longrightarrow & & L_{n-2}(\pi_2) & & \xrightarrow{\partial_2} \\
 & s\nearrow & & \searrow & \\
LP_{n-3}(F_2) & & \Gamma\downarrow & & L_{n-2}(\rho_2 \to \pi_2) \\
 & q\searrow & & \nearrow & \\
\longrightarrow & & L_{n-3}(\pi_3) & & \longrightarrow \\
 & & \vdots & &
\end{array} \tag{22}$$

In this diagram we denote by s and q all similar maps. But, in what follows, it will be clear from the context what map is under consideration. Note that the groups and the maps in diagram (22) are defined by the subscripts taken mod 4 since L_*-groups and diagram (20) are four periodic. But for our purposes we can consider arbitrary integer numbers as possible subscripts in diagram (22). Diagram (22) is infinite in bottom direction if and only if diagram (21) is infinite.

Now we can give an inductive definition of the sets

$$\Gamma^j(x) \subset L_{n-j}(\pi_j) \text{ for } (j \geq 0)$$

and iterated Browder-Livesay invariants with respect to diagram (21) $\mathcal{D}_k$ [7], [13], [14], [17]. For a finite k we suppose that $j \leq k$.

Definition 2.1 *Consider diagram (22) for a diagram $\mathcal{D}_k$. Let $x \in L_n(\pi_0)$. By definition,*

$$\Gamma^0(x) = \{x\} \subset L_n(\pi_0).$$

Let the set

$$\Gamma^j(x) \subset L_{n-j}(\pi_j) \quad (j \geq 0)$$

be defined. It is called trivial if $0 \in \Gamma^j(x)$.

If $\Gamma^j(x)$ is defined and nontrivial then the $(j+1)-th$ Browder-Livesay invariant with respect to $\mathcal{D}_k$ (21) is the set

$$\partial_j(\Gamma^j(x)) \subset LN_{n-j-2}(\rho_j \to \pi_j).$$

The $(j+1)-th$ invariant is nontrivial if $0 \notin \partial_j(\Gamma^j(x))$.

If the $(j+1)-th$ Browder-Livesay invariant is defined and trivial then the set $\Gamma^{j+1}(x)$ is defined as

$$\Gamma^{j+1}(x) \stackrel{def}{=} \{qs^{-1}(z) | z \in \Gamma^j(x), \partial_j(z) = 0\} \subset L_{n-j-1}(\pi_{j+1}). \tag{23}$$

Definition 2.2 *Let $\mathcal{D}_k$ be diagram (21) finite or infinite. An element $x \in L_n(\pi)$ has the second type with respect to $\mathcal{D}_k$ if the sets $\Gamma^j(x)$ are defined and nontrivial for $0 \leq j \leq k$.*

Note, that diagram (22) and Definitions 2.1 and 2.2 can be given for an arbitrary decoration U [6], [9], [15], [19]. The results of Kharshiladze about nonrealization of elements of the second type are based on consideration of infinite diagrams (22) of a special type, when all the inclusions have the form $\rho \to \pi^{\pm}$. Already in this very special case the algebraic properties of the elements of the second type are very interesting [1], [15]. Using diagram $\mathcal{D}_k$ we can algebraically define second type elements of the group $L_n^U(\pi_0)$. In the case of finite k, using four-periodicity of diagram (20), we can suppose that n is sufficiently large and $n - k \geq 5$.

3 Main Theorems

In this section we state and prove the main theorems of the paper. We give several sufficient conditions for nonrealization of the elements of the second type. These results jointly with Theorem 1.1 provide very effective methods of computations for finite 2-groups with decoration "prime". We use notations from the previous sections.

Theorem 3.1 *Let $x \in L_n^U(\pi_0)$ be an element of the second type with respect to diagram $\mathcal{D}_k$ for the group of decorations U. If for $l \leq k$, $l \in N$ there exists a number $m \geq 5$, $m \equiv n - l \pmod 4$ such that*

$$C_m^U(\pi_m) \bigcap \Gamma^l(x) = \emptyset \tag{24}$$

then the element x does not belong to the image of the Assembly map

$$A^U : H_n(B\pi_0, \boldsymbol{L}_\bullet) \to L^U_n(\pi_0).$$

In the case of finite 2-groups with decoration "prime" the results of Theorem 1.1 sometimes give us the possibility to avoid infinite diagrams $\mathcal{D}_k$ and consider only finite diagrams.

Theorem 3.2 *Let $\mathcal{D}_k$ consist of finite 2-groups and consider diagram (22) of the groups equipped with decoration "prime". Let $x \in L'_n(\pi_0)$ be an element of the second type with respect to diagram $\mathcal{D}_k$.*

i) If for $l \equiv n \bmod 4$ the set of numbers $\{l, l-1, \dots, l-k\}$ contains a number of type $4m+1$ such that the group π_{l-4m-1} has the trivial orientation, and the subset

$$\Gamma^{l-4m-1}(x) \subset L'_{4m+1}(\pi_{l-4m-1})$$

maps to zero under transfer-projection to all quaternion subquotients $Q(2^k)$ of the group π_{l-4m-1} then the element x does not belong to the image of the Assembly map

$$A' : H_n(B\pi_0, \boldsymbol{L}_\bullet) \to L'_n(\pi_0).$$

ii) If for $l \equiv n \bmod 4$ the set of numbers $\{l, l-1, \dots, l-k\}$ contains a number of type $4m+3$ such that the group π_{l-4m-3} has the trivial orientation, and the subset

$$\Gamma^{l-4m-3}(x) \subset L'_{4m+3}(\pi_{l-4m-3})$$

maps to zero under projection to all $Z/2$ quotients of the group π_{l-4m-3} then the element x does not belong to the image of the Assembly map

$$A' : H_n(B\pi_0, \boldsymbol{L}_\bullet) \to L'_n(\pi_0).$$

Theorem 3.3 *Let $\mathcal{D}_k$ consist of finite abelian 2-groups and consider diagram (22) of the groups equipped with decoration "prime". Let $x \in L'_n(\pi_0)$ be an element of the second type with respect to $\mathcal{D}_k$. If for $l \equiv n \bmod 4$ the set of numbers $\{l, l-1, \dots, l-k\}$ contains a number of type $4m+1$ and the group π_{n-4m-1} has the trivial orientation then the element x does not belong to the image of the Assembly map*

$$A' : H_n(B\pi_0, \boldsymbol{L}_\bullet) \to L'_n(\pi_0).$$

Note that using the development approach in many cases we can avoid difficult computations to obtain explicit results about the Assembly map not only in the oriented case. The following theorem illustrates this for elementary abelian 2-groups. This result is known [12], [14]. But we give here a very simple proof using only diagrams $\mathcal{D}_1$ and very easy computations (compare with [12], [14], and [16]). We recall that for an elementary 2-group π there exist only two possible orientations up to isomorphism of oriented groups. As usual let π denote the group with trivial orientation and π^- with nontrivial.

Theorem 3.4 *Let π be an elementary abelian 2-group of rank $r \geq 1$.*
i) For the groups

$$L_1(\pi) = (Z/2)^{2^r - r - 1 - \frac{r(r-1)}{2}}, \quad \textit{and} \quad L_{2k+1}(\pi^-) = (Z/2)^{2^{r-1}-r}$$

the Assembly map is trivial.
ii) For the groups

$$L_2(\pi) = Z/2, \quad \textit{and} \quad L_{2k}(\pi) = (Z/2)^{2^{r-1}-r+1}$$

the image of the Assembly map is $Z/2$. The nontrivial element is given by the Arf-invariant.
iii) For the group

$$L_0(\pi) = Z^{2^r} \oplus (Z/2)^{2^r - r - 1 - \frac{r(r-1)}{2}}$$

the image of the Assembly map is Z and it coincides with the image

$$Z = L_0(1) \stackrel{mono}{\longrightarrow} L_0(\pi)$$

of the map induced by the inclusion $1 \to \pi$.
iv) Let

$$A = (Z/2)^r \subset L_3(\pi)$$

be a subgroup of the group $L_3(\pi)$ generated by all images

$$Z/2 = L_3(1) \to L_3(\pi)$$

of all maps that induced by inclusions of $Z/2$ on the direct summand. For the group

$$L_3(\pi) = (Z/2)^{2^r - 1}$$

the image of the Assembly map coincides with the subgroup A.

Now we prove the Theorems stated above.

Proof of Theorem 3.1. The multiplication of the closed manifold surgery problem (1) with the complex projective plain CP^2 provides an inclusion $C_n^U(\pi) \to C_{n+4}^U(\pi)$ [23]. Thus, using the multiplication on CP^2 and the four-periodicity of diagram (20) we can suppose that $n - l = m \geq 5$ with n, l, m being natural numbers. Let the element $x \in L_n^U(\pi_0)$ be realized by a normal map (1) of closed manifolds. Taking a characteristic submanifold $X_1 \subset X$ with respect to inclusion $\rho_0 \to \pi_0$ and taking the restriction of the map f to the transversal preimage $M_1 = f^{-1}(X_1)$ we obtain a closed manifold surgery problem

$$(f_1, b_1) = (f\,|_{M_1}, b|_{M_1}) : (M_1^{n-1}, \nu_M \Big|_{M_1}) \to (X_1^{n-1}, \nu_X \Big|_{X_1}) \tag{25}$$

with an obstruction

$$x_1 = \sigma^U_{n-1}(f_1, b_1) \in L_{n-1}(\pi_1).$$

The maps s and q in diagram (20) are the natural forgetful maps, and the pair of maps (f, f_1) gives a normal map of manifold pairs with an obstruction in the group $LP^U_{n-1}(F_0)$ [19], [23]. Thus the element x_1 lies in the class $\Gamma^1(x)$ and it is represented by the normal maps of closed manifolds (25) (see also, [7], [13], [14], and [17]). Iterating this construction we obtain a closed manifold surgery problem

$$(f_l, b_l) : (M^m_l, \nu_M\,|_{M_l}) \to (X^m_l, \nu_X|_{X_l}) \tag{26}$$

with an obstruction

$$x_l = \sigma^U_m(f_l, b_l) \in L_{n-l}(\pi_l) = L_m(\pi_l).$$

Definition 2.1 of the sets $\Gamma^j(x_0)$ implies that $x_l \in \Gamma^l(x_0)$ and it is represented by a normal map of closed manifolds (26), that is $x_l \in C^U_m(\pi_l)$. We obtain the contradiction with (24) and Theorem 3.1 is proved.

Proofs of the other theorems are based on Theorem 1.1 and Theorem 3.1.

Proof of Theorem 3.2. Consider case i) of the theorem. Let

$$4m + 1 \in \{l, l-1, \dots, l-k\}.$$

As in the prove of Theorem 3.1, using multiplication with CP^2, we can suppose that $l - k \geq 5$ and hence $l \geq 4m + 1 \geq 5$. Let nontrivial element $x \in L_l(\pi_0) = L_n(\pi_0)$ be realized by a normal map of closed manifolds. By Theorem 3.1 this implies that

$$C^U_{4m+1}(\pi_{l-4m-1}) \bigcap \Gamma^{l-4m-1}(x) \neq \emptyset$$

and hence contains a nontrivial element that is realized by a normal map of closed manifolds. By Theorem 1.1 this element maps to a nonzero element under transfer-projection with respect to a quaternion subquotient $Q(2^k)$. We obtain a contradiction. Case ii) is similar and Theorem 3.2 is proved.

Proof of Theorem 3.3. Since π_0 is a finite abelian 2-group, the group π_{l-4m-1} is the finite abelian 2-group with trivial orientation, and it does not have the quaternion subquotients. As before we can suppose that $4m+1 \geq 5$. Hence, as follows from Theorem 1.1, any nontrivial element of the group $L'_{4m+1}(\pi_{l-4m-1})$ cannot be realized by a normal map of closed manifolds. By Definitions 2.1 and 2.2 the subset

$$\Gamma^{l-4m-1}(x) \subset L'_{4m+1}(\pi_{l-4m-1})$$

contains only nontrivial elements. Let an element x be realized by a normal map of closed manifolds. Using the same line of arguments as in the proofs of Theorems 3.1 and 3.2 we obtain a nontrivial element in $L'_{4m+1}(\pi_{l-4m-1})$ that

belongs to the image of the Assembly map. We obtain a contradiction and Theorem 3.3 is proved.

Proof of Theorem 3.4. At first we recall that for any elementary 2-group π, $Wh(\pi) = 0$ and $L'_*(\pi) = L^s_*(\pi)$.

Now the case $L_1(\pi)$ follows directly from Theorem 1.1 and the Arf-invariant is realized in the group $L_2(\pi)$ [23].

Let $i : \rho \to \pi^-$ be an inclusion of index 2 where ρ has the trivial orientation. Note that this inclusion gives a diagram $\mathcal{D}_1$ with $\pi_0 = \pi^-$ and $\pi_1 = \pi$. Consider the following part of diagram (20) for the inclusion i:

$$\begin{array}{ccccc} Z/2 = L_2(\rho) & \overset{mono}{\longrightarrow} & L_2(\pi^-) & \overset{0}{\longrightarrow} & LN_0(\rho \to \pi^-) \\ & \searrow \quad \nearrow & & \searrow \quad \nearrow & \\ & LP_1(F_0) & \Gamma \downarrow & L_2(\rho \to \pi^-) & \\ & \nearrow \quad \searrow & & \nearrow \quad \searrow & \\ LN_1(\rho \to \pi^-) & \overset{mono}{\longrightarrow} & L_1(\pi) & \overset{0}{\longrightarrow} & L_1(\rho) \end{array} \tag{27}$$

in which all the maps follow by diagram chasing. It follows immediately from diagram (27) that $\Gamma^1(x)$ is defined and nontrivial for all $x \in L_2(\pi^-)$ except the direct summand $Z/2$ which coincides with the image $L_2(1)$ by functoriality. Now the statement of the theorem for the group $L_2(\pi^-)$ follows from Theorem 3.3.

Let

$$i_- : \rho^- \to \rho^- \oplus Z/2^- = \pi^-$$

be an inclusion of index 2 of elementary 2-groups. Consider the following part of diagram (20) for i_-:

$$\begin{array}{ccccc} L_3(\rho^-) & \overset{mono}{\longrightarrow} & L_3(\pi^-) & \overset{0}{\longrightarrow} & LN_1(\rho^- \to \pi^-) \\ & \searrow \quad \nearrow & & \searrow \quad \nearrow & \\ & LP_2(F_0) & \Gamma \downarrow & L_3(\rho^- \to \pi^-) & \\ & \nearrow \quad \searrow & & \nearrow \quad \searrow & \\ LN_2(\rho^- \to \pi^-) & \overset{mono}{\longrightarrow} & L_2(\pi^-) & \overset{0}{\longrightarrow} & L_2(\rho^-). \end{array}$$

If

$$x \notin \mathrm{Im}\{L_3(\rho^-) \to L_3(\pi^-)\}$$

then $\Gamma^1(x)$ is defined and nontrivial, and hence

$$\Gamma^1(x) \bigcap \mathrm{Im}\{LN_2(\rho^- \to \pi^-) \to L_2(\pi^-)\} = \emptyset.$$

The image $\mathrm{Im}\{LN_2(\rho^- \to \pi^-) \to L_2(\pi^-)\}$ contains the direct summand $Z/2$ coming from $L_2(1)$. Thus, using the statement of the theorem for $L_2(\pi^-)$ and Theorem 3.1 we obtain that all the elements of the group $L_3(\pi^-)$ which do not belong to the image

$$\mathrm{Im}\{i_* : L_3(\rho^-) \to L_3(\pi^-)\}$$

cannot be realized by a normal map of closed manifolds. Let a nontrivial element

$$x \in \mathrm{Im}\{i_* : L_3(\rho^-) \to L_3(\pi^-)\}$$

be in the image of the Assembly map and $x = i_*(y), y \in L_3(\rho^-)$. The map i_* has the right inverse map

$$p_* : L_3(\pi^-) \to L_3(\rho^-)$$

since i_- is isomorphic to an inclusion of the direct summand. Thus $p_* i_*(y) = y$ and a nontrivial element y belongs to the image of the Assembly map for the group ρ^- of the rang $(r-1)$. Similar to the above this implies that y must belong to the image of the map $L_3(\rho_1^-) \to L_3(\rho^-)$ induced by inclusion $\rho_1^- \to \rho^-$, and $\mathrm{rang}(\rho_1) = r-2$. Iterating this process we obtain that the nontrivial element of the group $L_3(Z/2 \oplus Z/2 \oplus Z/2^-) = Z/2$ must belong to the image of the natural map

$$L_3(Z/2 \oplus Z/2^-) \to L_3(Z/2 \oplus Z/2 \oplus Z/2^-) = Z/2$$

which is trivial, since the first group is trivial. We obtain a contradiction, and hence the statement of the theorem for the group $L_3(\pi^-)$.

Now consider the following part of diagram (20) for an inclusion $i : \rho \to \pi$ of index 2 with $\mathrm{rang}(\pi) = r$:

$$\begin{array}{ccccc} L_0(\rho) & \overset{mono}{\longrightarrow} & L_0(\pi) & \overset{\partial}{\longrightarrow} & LN_2(\rho \to \pi) \\ & \searrow \quad \nearrow & & \searrow & \\ & LP_3(F_0) & \Gamma \downarrow & & L_0(\rho \to \pi) \\ & \nearrow \quad \searrow & & \nearrow & \\ LN_3(\rho \to \pi^-) & \overset{mono}{\longrightarrow} & L_3(\pi^-) & \overset{0}{\longrightarrow} & L_3(\rho). \end{array} \tag{28}$$

The map ∂ in diagram (28) is the first Browder-Livesay invariant and the elements that do not belong to the kernel of this map cannot be realized by normal maps of closed manifolds [5]. If an element $x \in \mathrm{Ker}\partial$ and

$$x \notin \mathrm{Im}\{L_0(\rho) \to L_0(\pi)\}$$

then $\Gamma^1(x)$ is defined and nontrivial. Thus

$$0 \notin \Gamma^1(x) \subset L_3(\pi^-)$$

and we already proved that all the nontrivial elements of $L_3(\pi^-)$ cannot be realized. By Theorem 3.1 we obtain that the element x does not belong to the image of the Assembly map. From now on going to the subgroup ρ, as in the case $L_3(\pi^-)$, we obtain that the image of the Assembly map for the group $L_0(\pi)$ coincides with the image

$$Z = L_0(1) \overset{mono}{\longrightarrow} L_0(\pi).$$

In the case $L_3(\pi)$, the subgroup $A \subset L_3(\pi)$ belongs to the image of the Assembly map by functoriality and by realization of the nontrivial element of the group $L_3(Z/2)$ [23]. Consider the diagram

$$\begin{array}{ccccccc} L_3(\rho) & \overset{mono}{\longrightarrow} & & L_3(\pi) & & \overset{\mathrm{Im}=(Z/2)^{r-1}}{\longrightarrow} & LN_1(\rho \to \pi) \\ & \searrow & & \nearrow & \searrow & & \\ & & LP_2(F_0) & & \Gamma \downarrow & & L_3(\rho \to \pi) \\ & \nearrow & & \searrow & & \nearrow & \\ LN_2(\rho \to \pi) & & \overset{0}{\longrightarrow} & & L_2(\pi^-) & \overset{0}{\longrightarrow} & L_3(\rho). \end{array} \tag{29}$$

The natural retraction R of the inclusion $\rho \to \pi^{\pm}$ to the inclusion $1 \to Z/2^{\pm}$ induces the retraction R_* of diagram (29) to the similar diagram for the inclusion $1 \to Z/2^{\pm}$ which has the following form

$$\begin{array}{ccccc} 0 = L_3(1) & \overset{0}{\longrightarrow} & \{L_3(Z/2) = Z/2\} & \overset{0}{\longrightarrow} & 0 \\ & \searrow & \nearrow & \searrow & \\ & Z/2 & & \Gamma \downarrow & Z/2 \\ & \nearrow & \searrow & \nearrow & \\ Z = LN_2(1 \to Z/2) & \overset{0}{\longrightarrow} & \{L_2(Z/2^-) = Z/2\} & \overset{0}{\longrightarrow} & Z/2. \end{array} \tag{30}$$

The map of diagrams R_* has the right inverse which is an inclusion on each group. Note that the nontrivial element of $L_2(Z/2^-)$ has nontrivial Arf-invariant and belongs to the image of the Assembly map. Hence as before, using the retraction R we obtain that only the elements belonging to the images of the map $i_* : L_3(\rho) \to L_3(\pi)$ and of the map $L_3(Z/2) \to L_3(\pi)$ can be in the image of the Assembly map. As before iterating this we obtain case iv) of the theorem.

The consideration in the case $L_0(\pi^-)$ is similar. Consider the diagram

$$\begin{array}{ccccccc} L_0(\rho) & \overset{0}{\longrightarrow} & & L_0(\pi^-) & & \overset{0}{\longrightarrow} & LN_2(\rho \to \pi^-) \\ & \searrow & & \nearrow & \searrow & & \\ & & LP_3(F_0) & & \Gamma \downarrow & & L_0(\rho \to \pi^-) \\ & \nearrow & & \searrow & & \nearrow & \\ LN_3(\rho \to \pi^-) & & \overset{mono}{\longrightarrow} & & L_3(\pi) & \overset{\mathrm{Im}=(Z/2)^{r-1}}{\longrightarrow} & L_3(\rho). \end{array} \tag{31}$$

The natural retraction R of the inclusions induces the retraction R_* of diagram (31) to the diagram for the inclusion $1 \to Z/2^{\pm}$ which has the following form

$$\begin{array}{ccccc} Z = L_0(1) & \overset{0}{\longrightarrow} & Z/2 & \overset{0}{\longrightarrow} & Z/2 \\ & \searrow & \nearrow & \searrow & \\ & Z/2 & \Gamma \downarrow & Z/2 & \\ & \nearrow & \searrow & \nearrow & \\ 0 = LN_3(1 \to Z/2^-) & \overset{0}{\longrightarrow} & Z/2 & \overset{0}{\longrightarrow} & 0. \end{array} \tag{32}$$

The nontrivial element of $L_0(Z/2^-)$ is the Arf-invariant and belongs to the image of the Assembly map. The image of the group $LN_3(\rho \to \pi^-)$ in the group $L_3(\pi)$ coincides with the image of the map $i_* : L_3(\rho) \to L_3(\pi)$ [16, §2]. Thus the image A of the Assembly map to the group $L_3(\pi)$ belongs to the subgroup generated by the image of the map $L_3(Z/2) \to L_3(\pi)$ that is induced by inclusion on the direct summand. It is given by the right inverse to the map R and by the image of the map $\{LN_3(\rho \to \pi^-) \longrightarrow L_3(\pi)\}$ from diagram (31). Now we obtain the statement of the theorem for the group $L_0(\pi^-)$ using the same line of arguments as above.

Now the consideration of the case $L_1(\pi^-)$ is similar to the case $L_3(\pi^-)$. Theorem 3.4 is proved.

4 Examples and Applications

In this section we consider several examples and applications of the obtained results. At first, we discuss an application to the problem of realization of elements of Browder-Quinn surgery obstruction groups by normal maps of closed manifolds. Consider a filtration $\mathcal{X}$

$$X_k \subset X_{k-1} \subset \cdots \subset X_2 \subset X_1 \subset X_0 = X \tag{33}$$

of X^n by locally flat embedded submanifolds. Let $\dim X_k \geq 5$. We suppose that every pair of manifolds $X_{j+1} \subset X_j$ is a Browder-Livesay pair with respect to the inclusion

$$i_j : \pi_1(X_j \setminus X_{j+1}) \to \pi_1(X_j) \tag{34}$$

and that every pair of manifolds from the given filtration is a topological manifold pair in the sense of Ranicki (see [19, §7.2]). Under these conditions, we shall call the filtration (5.1) *a Browder-Livesay filtration* [7], [13], [17]. Note that the filtration $\mathcal{X}$ provides a finite commutative diagram $\mathcal{D}_k$ with

$$\pi_j = \pi_1(X_j) \quad \text{and} \quad \rho_j = \pi_1(X_j \setminus X_{j+1})$$

and the maps from (34).

The Browder-Livesay filtration $\mathcal{X}$ is a stratified manifold in the sense of Browder-Quinn [4], [7], [17], [24]. The Browder-Quinn surgery obstruction groups are denoted by $L^{BQ}_{n-k}(\mathcal{X})$. Forgetting all manifolds of the filtration except a manifold X_j $(0 \leq j \leq k)$ gives natural maps [7], [17], [24]

$$p_j : L^{BQ}_{n-k}(\mathcal{X}) \to L_{n-j}(\pi_1(X_j)) = L_{n-j}(\pi_j). \tag{35}$$

Note that the decorated Browder-Quinn surgery obstruction groups are defined [6], [7], [17], [24]. We shall denote by

$$C^{BQ}_{n-k}(\mathcal{X}) \subset L^{BQ}_{n-k}(\mathcal{X})$$

the subgroup of obstructions generated by normal maps to a filtration of closed manifolds. The decorated group $C^{BQ^U}_{n-k}(\mathcal{X})$ is defined in a natural way.

It follows from the geometrical definition of this map that if an element $x \in L^{BQ}_{n-k}(\mathcal{X})$ is realized as an obstruction of a normal map to the closed filtered manifold X then for $0 \le j \le k$ the element $p_j(x) \in L_{n-j}(\pi_j)$ belongs to the image of the corresponding Assembly map. In fact, this statement is a geometrical interpretation of Theorem 2. The next results are easy corollaries from other theorems of section 3. In our consideration we fix a Browder-Livesay filtration $\mathcal{X}$ (33).

Corollary 4.1 *Let $\pi_0 = \pi_1(X_0)$ be a finite 2-group. Let $0 \ne x \in L^{BQ'}_{n-k}(\mathcal{X})$.*

i) If for $l \equiv n \bmod 4$ the set of numbers $\{l, l-1, \dots, l-k\}$ contains a number of type $4m+1$, the group π_{l-4m-1} has the trivial orientation, and the element

$$p_{l-4m-1}(x) \in L'_{4m+1}(\pi_{l-4m-1})$$

is nontrivial and maps to zero under transfer-projection to all quaternion subquotients $Q(2^k)$ of the group π_{l-4m-1} then the element x does not belong to the subgroup $C^{BQ'}_{n-k}(\mathcal{X})$.

i) If for $l \equiv n \bmod 4$ the set of numbers $\{l, l-1, \dots, l-k\}$ contains a number of type $4m+3$, the group π_{l-4m-3} has the trivial orientation, and the element

$$p_{l-4m-3}(x) \in L'_{4m+3}(\pi_{l-4m-3})$$

is nontrivial and maps to zero under projection to all $Z/2$ quotients of the group π_{l-4m-3} then the element x does not belong to the subgroup $C^{BQ'}_{n-k}(\mathcal{X})$.

Corollary 4.2 *Let $\pi_0 = \pi_1(X_0)$ be a finite abelian 2-group. Let $0 \ne x \in L^{BQ'}_{n-k}(\mathcal{X})$. If for $l \equiv n \bmod 4$ the set of numbers $\{l, l-1, \dots, l-k\}$ contains a number of type $4m+1$, the group π_{n-4m-1} has the trivial orientation, and*

$$p_{l-4m-1}(x) \in L'_{4m+1}(\pi_{l-4m-1})$$

is a nontrivial element then x does not belong to the subgroup $C^{BQ'}_{n-k}(\mathcal{X})$.

Corollary 4.3 *Let $\pi_0 = \pi_1(X_0)$ be an elementary 2-group. Let $0 \ne x \in L^{BQ}_{n-k}(\mathcal{X})$.*

If for $l \equiv n \bmod 4$ the set of numbers $\{l, l-1, \dots, l-k\}$ contains a number of type $4m+1$, and

$$p_{l-4m-1}(x) \in L_{4m+1}(\pi_{l-4m-1})$$

is a nontrivial element, or the set of numbers $\{l, l-1, \dots, l-k\}$ contains a number $4m+3$, the group π_{l-4m-3} has the nontrivial orientation, and

$$p_{l-4m-3}(x) \in L_{4m+3}(\pi_{l-4m-3})$$

is a nontrivial element then x does not belong to the subgroup $C^{BQ}_{n-k}(\mathcal{X})$.

It is fairly easy to generalize the results given above in this section to the case of an arbitrary filtration.

Now we give two examples of computations.

Example 4.4 *Let π^- be a finite abelian 2-group of the rank r and with $s \geq 1$ direct summands $Z/2$ with the nontrivial homomorphism orientation on a direct summand $Z/2$ (exceptional case [22]). Then [22, page 37]*

$$L'_2(\pi^-) = (Z/2)^{2^{r-1}-s+1}.$$

But $C'_2(\pi^-) = Z/2$, in fact it coincides with the image

$$\alpha : Z/2 = L_2(1) \to L'_2(\pi^-).$$

To see this, we consider diagram that is similar to diagram (27) for the inclusion

$$i : \rho^+ \to \rho^+ \oplus Z/2^- = \pi^-.$$

The diagram chasing implies that the induced map $i_* : L_2(\rho) \to L_2(\pi^-)$ has the image $Z/2$ that coincides with the image of α [16]. The elements of this image are in $C'_2(\pi^-)$ by functoriality. The map Γ is nontrivial for other elements. Now it is sufficient to apply Theorem 3.3.

Example 4.5 *Let $F = (Z)^r$ be a free abelian group of rank $r \geq 1$ and $\pi = (Z/2)^r$ be an elementary abelian 2-group with the same rank. Consider the canonical projection $p : F \to \pi$. Then comparing the results about the groups $C_*(F) = L_*(F)$ [22] and $C_*(\pi)$ from Theorem 3.4 and using functoriality [23] we obtain the full description of the induced map*

$$p_{*_n} : L_n(F) \to L_n(\pi).$$

The map p_{*_n} has image $Z,\ 0,\ Z/2,\ (Z/2)^r$ for $n = 0, 1, 2, 3 \bmod 4$, respectively. Such types of computations are very helpful for computations using diagram (20).

ACKNOWLEDGEMENTS. The authors thank the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, where the paper was done, for the hospitality and support. The second author is partially supported by the RFFI grant No. 05–01–00993; he thanks the Commission on Development an Exchange of the International Mathematical Union for the travel support.

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Received: April 2, 2007