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A Systematic Study of Frame Sequence Operators and their Pseudoinverses

P. Balazs

Acoustics Research Institute
Austrian Academy of Sciences, Vienna, Austria
Peter.Balazs@oeaw.ac.at

M. A. El-Gebeily

Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran, Saudi Arabia

Abstract

In this note we investigate the operators associated with frame sequences in a Hilbert space H, i.e., the synthesis operator $T : \ell^2(\mathbb{N}) \to H$, the analysis operator $T^* : H \to \ell^2(\mathbb{N})$ and the associated frame operator $S = TT^*$ as operators defined on (or to) the whole space rather than on subspaces. Furthermore, the projection P onto the range of T, the projection Q onto the range of T^* and the Gram matrix $G = T^*T$ are investigated. For all these operators, we investigate their pseudoinverses, how they interact with each other, as well as possible classification of frame sequences with them. For a tight frame sequence, we show that some of these operators are connected in a simple way.

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1 Introduction

Frame sequences are the natural generalization of frames [5]. In many situations, for example, when constructing a frame multi-resolution analysis (see e.g., [3], [8]), we start with a frame sequence in a Hilbert space H and then define the initial approximation space V as the closure of the span of the frame sequence. In the literature, several operators are associated with frame sequences and these spaces, namely, the projection operator $P : H \to H$ onto V, the inclusion operator $\iota_V : V \to H$, the analysis operator $\mathcal{U} : V \to \ell^2(\mathbb{N})$, the synthesis operator $\mathcal{T} : \ell^2(\mathbb{N}) \to V$ and the frame operator $\mathcal{S} : V \to V$ (the definition of all these operators will be given in the next section). In the literature, frame sequences are analyzed mostly using the concrete representation of these operators. On the other hand, analyzing frame sequences from a pure operator theoretic point of view can offer a deeper insight into the structure of such sequences. This note will take this approach. Almost exclusively, all the proofs provided in this paper use these operators and operator theoretic principles.

An alternative way of looking at the above operators is to extend their definitions with the help of the first two operators to the whole space H. This way we are always working with the base spaces H and $\ell^2(\mathbb{N})$ and we do not worry about the subspace V and its image in $\ell^2(\mathbb{N})$. The extended operators become: the synthesis operator $T : \ell^2(\mathbb{N}) \to H$, the analysis operator $U : H \to \ell^2(\mathbb{N})$ and the frame operator $S : H \to H$. We develop relationships between these extended versions. In fact, we start by defining these extended versions rather than the "classical" restricted operators. Consequently no ambiguity arises regarding notions such as inverses of operators and pseudo inverses. While the proofs of the relationships are straightforward in the most part, they are nontrivial in the sense that we use the current state of knowledge to derive them. Also, a form of duality between statements about the synthesis and the analysis operators emerges throughout the presentation. For all involved operators, we investigate how they interact with each other, as well as possible classification of frame sequences with them. For a tight frame sequence, we will show that some of these operators are connected in a simple way.

Of course, most of the work in the literature on frame sequences is related to this work. We mention in particular the references [7], [9], [12] which have more direct bearing on this note.

As a preliminary lemma we list here the properties of the pseudo-inverse or the Moore Penrose inverse of a bounded operator with closed range that are most important to us (see, for example, Appendix A.7 in [8]).

Lemma 1.1 *Let H_1, H_2 be Hilbert spaces, and suppose that $U : H_1 \to H_2$ is a bounded operator with closed range R_U. Then there exists a bounded operator $U^\dagger : H_2 \to H_1$ such that*

$$UU^\dagger f = f, \ \forall f \in R_U,$$

with $\ker_{U^\dagger} = \overline{R}_U^\perp$ and $\overline{R}_{U^\dagger} = \ker_U^\perp$. This operator is uniquely determined by these properties.

Furthermore, $U^\dagger$ has the following properties.

1. *$UU^\dagger$ is the orthogonal projection of H_2 onto R_U.*

2. *$U^\dagger U$ is the orthogonal projection of H_1 onto $R_{U^\dagger}$.*

3. *U^* has closed range and $(U^*)^\dagger = \left(U^\dagger\right)^*$.*

4. *$U^\dagger U U^\dagger = U^\dagger$.*

5. *On $\overline{R}_U$ we have $U^\dagger = U^* \left(UU^*\right)^{-1}$.*

2 Frame Sequences and Their Pseudoinverses

In this section H denotes a general Hilbert space and $\ell^2(\mathbb{N})$ denotes the space of absolutely square summable sequences of complex numbers. We will denote elements of $\ell^2(\mathbb{N})$ by lower case letters such as $c, d, \cdots$ etc. and, when we want to explicitly use the terms of the sequences $c, d, \cdots$, we will use Greek letters such as $(\zeta_k)_{k=1}^{\infty}, (\eta_k)_{k=1}^{\infty}, \cdots$ etc. When no confusion arises we will write these sequences as $(\zeta_k), (\eta_k), \cdots$ etc. We denote by $\{\epsilon_k\}_{k=1}^{\infty}$ the sequence of standard basis elements in $\ell^2(\mathbb{N})$.

Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence in H. With $\{f_k\}_{k=1}^{\infty}$ we associate three, possibly unbounded [1], operators: the synthesis operator $T : \ell^2(\mathbb{N}) \to H$ defined by? $Tc = \sum_{k=1}^{\infty} \zeta_k f_k$,, the analysis operator $U : H \to \ell^2(\mathbb{N})$ defined by $Uf = (\langle f, f_k \rangle)$ and the *frame operator* S defined by $Sf = TUf = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k$, whenever the right hand sides of these definitions exist. Observe that T is densely defined as its domain D_T contains all finite sequences (sequences which, eventually, consist of zeros) in $\ell^2(\mathbb{N})$. This implies that T has a well defined adjoint $T^* : \ell^2(\mathbb{N}) \to H$, which is a closed operator (see [11]). We also have

(A) $\ker_{T^*}$ is closed.

(B) $\ker_{T^*} = (R_T)^{\perp} = \left(\overline{R}_T\right)^{\perp}$.

(C) $H = \overline{R}_T \oplus \left(\overline{R}_T\right)^{\perp} = \overline{R}_T \oplus \ker_{T^*}$.

It follows that the orthogonal projection P of H onto $\overline{R}_T = (\ker_{T^*})^{\perp}$ is always well defined. The following lemma and its corollary are straightforward.

Lemma 2.1 $\operatorname{span}\{f_k\}_{k=1}^{\infty} \subseteq R_T \subseteq \overline{\operatorname{span}}\{f_k\}_{k=1}^{\infty}$.

Corollary 2.2 *If T has closed range, then $R_T = \overline{\operatorname{span}}\{f_k\}_{k=1}^{\infty}$.*

Recall that $\{f_k\}_{k=1}^{\infty}$ is a frame sequence in H if there are positive constants A, B such that

$$A\,\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\,\|f\|^2 \;\; \forall f \in \operatorname{span}\{f_k\}_{k=1}^{\infty}. \tag{1}$$

The frame sequence is tight if $A = B$. By the definition above, it becomes clear that it is a frame for $V = \overline{\operatorname{span}}\{f_k\}$. The restricted versions of the above operators are defined in the same way with the only difference that they work from or to V. We let $\iota_V : V \to H$ be the inclusion operator $\iota_V(f) = f$, $\mathcal{U} : V \to \ell^2(\mathbb{N})$ the analysis operator, $\mathcal{T} : \ell^2(\mathbb{N}) \to V$ the synthesis and $\mathcal{S} : V \to V$ the frame operator. We have following basic relationships between these operators, which are straightforward to show

Proposition 2.3 *If $\{f_k\}_{k=1}^{\infty}$ is a frame sequence in H, then the following properties hold.*

1. $T = \iota_V \mathcal{T}$.
2. $\overline{R}_T = \overline{R}_{\mathcal{T}} = V$ *and P is the projection on V.*
3. $U = \mathcal{U}P$.
4. $R_U = R_{\mathcal{U}}$.
5. $S = TU = \iota_V \mathcal{T}\mathcal{U}P = \iota_V \mathcal{S}P$.

Proof. (iii) & (Iv) : For $h_2 \in V^{\perp}$, we clearly have $U(h_2) = (\langle h_2, f_k\rangle) = 0$. Every $h \in H$ can be uniquely be described as $h = h_1 + h_2$ with $h_1 \in V$ and $h_2 \in V^{\perp}$, therefore, $U(h) = U(h_1) + U(h_2) = U(h_1) = \mathcal{U}Pf$.

All the others proofs are straightforward. ■

As a frame sequence is a frame for its closed span, $\mathcal{U}$ and $\mathcal{T}$ are bounded, $\mathcal{T} = \mathcal{U}^*$ and $\mathcal{U} = \mathcal{T}^*$. Also

Corollary 2.4 *If $\{f_k\}_{k=1}^{\infty}$ is a frame sequence, then*

1. *the analysis operator U is bounded,*
2. *the synthesis operator T is bounded,*
3. $T = U^*$ *and* $U = T^*$.

Proof. (i) and (ii): Since $\mathcal{T}$, $\mathcal{U}$, P and ι_V are bounded, the boundedness of U and T follows directly from the above relations.

(iii): $U^* = (\mathcal{U}P)^* = P^*\mathcal{U}^* = \iota_V \mathcal{T} = T$. Repeat the argument for T^*. ■

As $\{f_k\}$ is a frame for V, there is a sequence $\{\tilde{f}_k\} \subseteq V$, which is the canonical dual frame for V, $\tilde{f}_k = \mathcal{S}^{-1} f_k$. $\{\tilde{f}_k\}$ is again a frame sequence in H with closed span V and the bounds $\tilde{A} = \frac{1}{B}$ and $\tilde{B} = \frac{1}{A}$. Let $\tilde{T}, \tilde{U}, \tilde{S}, \tilde{\mathcal{T}}, \tilde{\mathcal{U}}$ and $\tilde{\mathcal{S}}$ be the corresponding operators associated with this frame sequence. Therefore we have $\tilde{\mathcal{T}} = \mathcal{S}^{-1}\mathcal{T}$, $\tilde{\mathcal{U}} = \mathcal{U}\mathcal{S}^{-1}$, $\tilde{\mathcal{S}} = \mathcal{S}^{-1}$ and $\mathcal{T}\tilde{\mathcal{U}} = \tilde{\mathcal{T}}\mathcal{U} = id_V$. The following corollary can be easily shown.

Corollary 2.5 *If $\{f_k\}_{k=1}^{\infty}$ is a frame sequence in H and $\{\tilde{f}_k\}$ is its dual sequence, then the following properties hold.*

1. $\tilde{T} = i_V \tilde{\mathcal{T}}$ *and* $\tilde{U} = \tilde{\mathcal{U}}P$.
2. $\tilde{S} = \iota_V \mathcal{S}^{-1} P$.
3. $T\tilde{U} = \iota_V P = \tilde{T}U$.

It follows from Property (3) above that the projection P on V as a function from H into H is $T\tilde{U}$. This is well known [8]. Also, denoting by Q the orthogonal projection of $\ell^2(\mathbb{N})$ onto $(\ker_T)^{\perp} = \overline{R}_{T^*}$, it is straightforward to show that Q is the Gram matrix $G = U\tilde{T} = \mathcal{U}\tilde{\mathcal{T}}$.

We have the following characterizations of frame sequences.

Theorem 2.6 *The following are equivalent:*

1. *$\{f_k\}_{k=1}^{\infty}$ is a frame sequence in H with bounds A, B.*
2. *There exist positive constants A, B such that for every $f \in H$,*
$$A \|Pf\|^2 \leq \|T^* f\|^2 \leq B \|Pf\|^2 .$$
3. *There exist positive constants A, B such that for every $c \in \ell^2(\mathbb{N})$,*
$$A \|Qc\|^2 \leq \|Tc\|^2 \leq B \|Qc\|^2 .$$

Proof. Assume (i) holds. As $T^* = U = \mathcal{U}P$, (ii) is equivalent to the definition of frame sequences and is therefore true.

Assume (ii) holds. Then a similar inequality holds for the dual frame in V, i.e.,

$$\tilde{A} \|Pf\|^2 \leq \left\|\tilde{U} f\right\|^2 \leq \tilde{B} \|Pf\|^2 , \text{ or } \frac{1}{B} \|Pf\|^2 \leq \left\|\tilde{U} f\right\|^2 \leq \frac{1}{A} \|Pf\|^2 .$$

Now choose $c \in \ell^2(\mathbb{N})$ and set $f = Tc$. Then $\frac{1}{B}\|PTc\|^2 \le \left\|\tilde{U}Tc\right\|^2 \le \frac{1}{A}\|PTc\|^2$, which implies that

$$\frac{1}{B}\|Tc\|^2 \le \|Qc\|^2 \le \frac{1}{A}\|Tc\|^2, \text{ or } A\|Qc\|^2 \le \|Tc\|^2 \le B\|Qc\|^2.$$

Assume (iii) holds. Since $\|Qc\| \le \|c\|$, it follows that T is continuous. Let $c \in \ker_T^\perp$. Then

$$A\|c\|^2 \le \|Tc\|^2 \le B\|c\|^2.$$

Therefore $T|_{\ker_T^\perp}$ is bounded, injective and has closed range [10, 2]. As $R_T = R_{T|_{\ker_T^\perp}}$, T is bounded and has closed range. Using [6] this is equivalent to $\{f_k\}$ forming a frame sequence. ■

It follows easily from Theorem 2.6 that $\{f_k\}_{k=1}^\infty$ is a frame for H if and only if T is surjective (i.e., $P = I$) and that $\{f_k\}_{k=1}^\infty$ is a Riesz Basis if and only if T^* is onto (i.e., $Q = I$). Theorem 2.6 can also be restated in a number of other ways; the following one uses the optimal bounds for the inequalities.

Corollary 2.7 *The following are equivalent:*

1. $\{f_k\}_{k=1}^\infty$ *is a frame sequence in* H.
2. T^* *is continuous, has a closed range, and*

$$\left\|T^\dagger\right\|^{-1}\|Pf\| \le \|T^*f\| \le \|T\|\,\|Pf\| \;\; \forall f \in H.$$

3. T *is continuous, has a closed range, and*

$$\left\|T^\dagger\right\|^{-1}\|Qc\| \le \|Tc\| \le \|T\|\,\|Qc\| \;\; \forall c \in \ell^2(\mathbb{N}).$$

For a frame sequence the frame operator $S = TT^*$ is bounded and self adjoint. Furthermore, since R_{T^*} is closed,

$$\begin{aligned} R_S &= TT^*H = T(T^*H + \ker_T) \\ &= T\left(T^*H + R_{T^*}^\perp\right) = T\left(R_{T^*} + R_{T^*}^\perp\right) \\ &= T\ell^2(\mathbb{N}) = R_T. \end{aligned}$$

It follows that S has closed range and, hence, a continuous Moore-Penrose pseudo-inverse $S^\dagger$. We list some properties of $S^\dagger$ next.

Lemma 2.8 *The operator* $S^\dagger : H \to H$ *is the same as the operator* $\tilde{S} = i_V \mathcal{S}^{-1} P$ *and therefore is self-adjoint and has the following properties:*

1. $SS^\dagger = S^\dagger S = P$.

2. $S^\dagger (I - P) = 0$.

3. $S^\dagger P = PS^\dagger = S^\dagger$.

Proof. $\tilde{T}$ has the same range as T as mentioned above, $\overline{R}_{\tilde{T}} = V$. Therefore, $\overline{R}_{\tilde{S}} = \overline{R}_S$. Also, $S\tilde{S} = \iota_V \mathcal{S} P \iota_V \tilde{\mathcal{S}} P = \iota_V \mathcal{S}\tilde{\mathcal{S}} P = \iota_V \mathcal{S}\mathcal{S}^{-1} P = \iota_V P$. Furthermore, $\ker \tilde{S} = \left\{ f : \tilde{S} f = 0 \right\} = \{ f : \iota_V \mathcal{S} P f = 0 \} = \{ f : Pf = 0 \} = V^\perp$, as ι_V and $\mathcal{S}$ are injective. Repeat the same argument for the frame sequence $\left\{ \widetilde{f_k} \right\}$ with the roles of S and $\tilde{S}$ switched and use Lemma 1.1 to arrive at $S^\dagger = \tilde{S}$.

(i) By Property (i) of Lemma 1.1, $SS^\dagger$ is the orthogonal projection onto $R_S = R_T$. Therefore, $SS^\dagger = P$. Switch the roles of S and $\tilde{S}$ to show the second part.

(ii) $S^\dagger (I - P) = S^\dagger - S^\dagger P = S^\dagger - S^\dagger S S^\dagger = S^\dagger - S^\dagger = 0$.

(iii) The equality $S^\dagger P = S^\dagger$ follows form (ii). To show that $PS^\dagger = S^\dagger$ observe first that $S^\dagger P : H \to R_S$. Hence, by 2, Lemma 1.1 and (1.), $S^\dagger = S^\dagger P = PS^\dagger P = PS^\dagger S S^\dagger = PS^\dagger$. ■

Because of the frame property on V, among all sequences $c \in \ell^2(\mathbb{N})$ which synthesize an $f \in H$, the sequence $c_0 = \left(\left\langle f, S^\dagger f_k \right\rangle \right)$ is the one with the minimum norm.

Similarly, among all elements $f \in H$ which analyze to a $c \in \ell^2(\mathbb{N})$, the element $f_0 = S^\dagger T c = \sum_{k=1}^{\infty} \zeta_k S^\dagger f_k$ is the one with the minimum norm. We have $\|f\|^2 = \|f_0\|^2 + \|f - f_0\|^2$.

Proposition 5.3.5 in [8] can now be restated in terms of $S^\dagger$ which is defined on all of H instead of $\mathcal{S}^{-1}$ which is defined only on V.

Corollary 2.9 *Let $\{f_k\}_{k=1}^{\infty}$ be a frame sequence in H . For any $f \in H$,*

$$Pf = \sum_{k=1}^{\infty} \left\langle f, S^\dagger f_k \right\rangle f_k.$$

Proposition 2.10 *The pseudo-inverse of T is $\tilde{U}$, $T^\dagger = \tilde{U}$. The pseudo-inverse of U is $\tilde{T}$, $U^\dagger = \tilde{T}$. Consequently, we have the following properties*

1. $T^\dagger = T^* S^\dagger$ *and* $T^\dagger = S^\dagger T^*$

2. $\left(T^\dagger\right)^* T^\dagger = S^\dagger$.

3. $\left(T^\dagger\right)^* = S^\dagger T$.

4. $\|T\|^2 = \|S\|$.

5. $\left\|T^{\dagger}\right\|^2 = \left\|S^{\dagger}\right\|$.

Proof. Clearly $T\tilde{U} = \iota_V P$ and $ker_{\tilde{U}} = V^{\perp}$. Again by switching the roles of T and $\tilde{T}$ we arrive with Lemma 1.1 at $T^{\dagger} = \tilde{U}$. Use an analog argument for U and $\tilde{U}$.

(i) $\tilde{U} = \iota_V \widetilde{\mathcal{U}} P = \iota_V \ \widetilde{\mathcal{U}} \left(\tilde{\mathcal{T}}\mathcal{U}\right) P = \iota_V \ \left(\widetilde{\mathcal{U}}\tilde{\mathcal{T}}\right) \mathcal{U} P = \iota_V \ \tilde{\mathcal{S}}\mathcal{U} P = \iota_V \ \tilde{\mathcal{S}} P \iota_V \mathcal{U} P = \tilde{S} U$, so $T^{\dagger} = S^{\dagger} T^*$. Since $P = SS^{\dagger} = S\tilde{S} = (TT^*)\left(\tilde{T}\tilde{T}^*\right) = T \cdot \left[T^*(TT^*)^{\dagger}\right]$, it follows from the uniqueness of the Moor-Penrose pseudo-inverse that $T^{\dagger} = T^* S^{\dagger}$.

(ii) This follows immediately from $T^{\dagger} = \tilde{U}$

(iii) follows from (i) by taking conjugates.

(iv): We have for every $f \in H$, $\|T^* f\|^2 = \langle Sf, f\rangle \leq \|Sf\| \|f\| \leq \|S\| \|f\|^2$ .On the other hand, $\|S\| = \|TT^*\| \leq \|T\|^2$.

(v): Observe first that, for every $f \in H$,

$$\left\|T^{\dagger} f\right\|^2 = \left\langle f, S^{\dagger} f\right\rangle. \tag{2}$$

This can be seen as follows:

$$\begin{aligned} \left\|T^{\dagger} f\right\|^2 &= \left\langle T^* S^{\dagger} f, T^* S^{\dagger} f\right\rangle = \left\langle TT^* S^{\dagger} f, S^{\dagger} f\right\rangle \\ &= \left\langle Pf, S^{\dagger} f\right\rangle = \left\langle f, PS^{\dagger} f\right\rangle = \left\langle f, S^{\dagger} f\right\rangle. \end{aligned}$$

Hence, for every $f \in H$, $\left\|T^{\dagger} f\right\|^2 \leq \left\|S^{\dagger}\right\| \|f\|^2$ .It follows that $\left\|T^{\dagger}\right\|^2 \leq \left\|S^{\dagger}\right\|$ .On the other hand, since $S^{\dagger}$ is self adjoint,

$$\left\|S^{\dagger}\right\| = \sup_{\|f\|=1} \left\langle S^{\dagger} f, f\right\rangle = \sup_{\|f\|=1} \left\|T^{\dagger} f\right\|^2 \leq \left\|T^{\dagger}\right\|^2.$$

Therefore, $\left\|S^{\dagger}\right\| = \left\|T^{\dagger}\right\|^2$. ■

Let $\{f_k\}_{k=1}^{\infty}$ be a frame sequence in H. Define the Gram matrix $G : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $G = UT = T^*T$.Alternatively, $G = \mathcal{U} P i_V \mathcal{T} = \mathcal{U}\mathcal{T}$. More explicitly, $Gc = \sum_{l=1}^{\infty} \left(\sum_{k=1}^{\infty} c_k \langle f_k, f_j\rangle\right) \epsilon_j$ and $\langle Gc, c\rangle = \sum_{j,k=1}^{\infty} \langle c_k f_k, c_j f_j\rangle$ .It immediately follows from the definitions that $T^*S = GT^*$ and $ST = TG$. Clearly G is self adjoint. It is well known [2, 8] that G is a bijective bounded operator from R_{T^*} onto R_T with bounded inverse if and only if $\{f_k\}$ is a frame sequence. In particular $R_G = R_T$ and $\ker_G = \ker_T = R_{T^*}^{\perp}$. It follows that G has a closed range and, hence, a continuous Moor-Penrose pseudo-inverse $G^{\dagger}$. The Gram matrix is the projection onto $(\ker_T)^{\perp} = \overline{R}_{T^*}$.

We list some properties of $G^{\dagger}$ next. For that let us denote by $\tilde{G}$ the Gram matrix corresponding to the dual frame $\left\{\widetilde{f_k}\right\}$.The proof is analogous to the one of Lemma 2.8 with appropriate adjustments.

Lemma 2.11 *The operator $G^{\dagger}:\ell^2(\mathbb{N})\to\ell^2(\mathbb{N})$ is the same as $\tilde{G}$. It is therefore self-adjoint and has the following properties:*

1. $GG^{\dagger}=Q=G^{\dagger}G$.
2. $G^{\dagger}(I-Q)=0$.
3. $G^{\dagger}Q=QG^{\dagger}=G^{\dagger}$.

The following corollary is the same as [8], Proposition 5.3.6.

Corollary 2.12 *Let $\{f_k\}_{k=1}^{\infty}$ be a frame sequence in H . For any $c\in\ell^2(\mathbb{N}), Qc=\sum_{k=1}^{\infty}\left\langle c,G^{\dagger}T^{*}f_k\right\rangle\epsilon_k$.*

We may also write $Qc=\sum_{k,j=1}^{\infty}\left\langle c,G^{\dagger}\epsilon_j\right\rangle\langle f_j,f_k\rangle\epsilon_k$.

Lemma 2.13 *We have the following properties*

1. $(T^{*})^{\dagger}=TG^{\dagger}$.
2. $T^{\dagger}\left(T^{\dagger}\right)^{*}=G^{\dagger}$.
3. $T^{\dagger}=G^{\dagger}T^{*}$.
4. $\|T\|^2=\|G\|$.
5. $\left\|T^{\dagger}\right\|^2=\left\|G^{\dagger}\right\|$.

Proof. (i): Since $Q=T^{*}TG^{\dagger}=T^{*}(T^{*})^{\dagger}$, it follows from the uniqueness of the Moor-Penrose pseudo-inverse that $(T^{*})^{\dagger}=TG^{\dagger}$.

(ii) follows immediately from (1.) by multiplying on the left by $T^{\dagger}$ and using Property 3 of Lemma 1.1.

(iii) follows from (1.) by taking adjoints.

(iv): We have for every $c\in\ell^2(\mathbb{N})$, $\|Tc\|^2=\langle Gc,c\rangle\leq\|G\|\|c\|^2$. Thus $\|T\|^2\leq\|G\|$ .On the other hand, $\|G\|=\|T^{*}T\|\leq\|T\|^2$, which yeilds $\|G\|=\|T\|^2$.

(v): As $T^{\dagger}=\tilde{T}$ and $G^{\dagger}=\tilde{G}$, (5) is the same as (4) for the dual frame. ■

Corollary 2.14 $\|G\|=\|S\|$ *and* $\left\|G^{\dagger}\right\|=\left\|S^{\dagger}\right\|$.

Theorem 2.6 (or rather, Corollary 2.7) can also be reformulated as

Theorem 2.15 *The following are equivalent:*

1. *$\{f_k\}_{k=1}^{\infty}$ is a frame sequence in H.*

2. *S is continuous, has closed range and*

$$\left\|S^{\dagger}\right\|^{-1}\|Pf\|^2 \leq \langle Sf, f\rangle \leq \|S\|\,\|Pf\|^2 \ \ \forall f \in H.$$

3. *G is continuous, has closed range and*

$$\left\|S^{\dagger}\right\|^{-1}\|Qc\|^2 \leq \langle Gc, c\rangle \leq \|S\|\,\|Qc\|^2 \ \ \forall c \in \ell^2\left(\mathbb{N}\right).$$

Lemma 2.16 *Let* $\{f_k\}_{k=1}^{\infty}$ *be a tight frame sequence in* H*. Then*

1. $S = AP,\ G = AQ,$
2. $S^{\dagger} = \frac{1}{A}P,\ G^{\dagger} = \frac{1}{A}Q.$

Proof. We prove the statements for G only. Since the frame is tight, $\langle Gc, c\rangle = \|Tc\|^2 = A\,\|Qc\|^2 = A\,\langle Qc, Qc\rangle$. ■

An analog of the polarization identity can be easily proved:

$$\langle Gc, d\rangle = \begin{array}{l} \frac{1}{4}\left(\langle\, G\left(c+d\right), c+d\rangle - \langle\, G\left(c-d\right), c-d\rangle + \right. \\ \left. +i\,\langle\, G\left(c+id\right), c+id\rangle - i\,\langle\, G\left(c-id\right), c-id\rangle\right), \end{array}$$

which yields

$$\begin{aligned} \langle Gc, d\rangle &= \begin{array}{l} \frac{1}{4}A\left(\ \|Q\left(c+d\right)\|^2 - \|Q\left(c-d\right)\|^2 + \right. \\ \left. +i\,\|Q\left(c+id\right)\|^2 - i\,\|Q\left(c-id\right)\|^2\right) \end{array} \\ &= A\,\langle Qc, Qd\rangle = A\,\langle Qc, d\rangle\,. \end{aligned}$$

Since d is arbitrary, $Gc = AQc$. Furthermore, $Qc = G^{\dagger}Gc = G^{\dagger}AQc = AG^{\dagger}c$. This gives $G^{\dagger}c = \frac{1}{A}Qc$.

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