International Mathematical Forum, 3, 2008, no. 43, 2139 - 2147

New Hardy-Hilbert's-Type Integral Inequalities

W. T. Sulaiman

Dept. of Mathematics, College of Computer Sciences and Mathematics
University of Mosul, Mosul, Iraq

Abstract

Several new types of integral inequalities similar to Hardy-Hilbert's integral inequality are obtained.

Keywords: Hardy-Hilbert's inequality, integral inequality

1. Introduction

If $f, g \geq 0$ such that

$$0 < \int_0^\infty f^2(x)\,dx < \infty, \quad 0 < \int_0^\infty g^2(x)\,dx < \infty ,$$

then the famous Hilbert's integral inequality is given by

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dx\,dy < \pi\left(\int_0^\infty f^2(x)\,dx \int_0^\infty g^2(x)\,dx\right)^{1/2}. \tag{1}$$

where the constant factor π is the best possible (see[2]) . Inequality (1) has been generalized by Hardy-Riesz [1] as

If p>1, $\frac{1}{p}+\frac{1}{q}=1$,

$$0 < \int_0^\infty f^p(x)\,dx < \infty , \quad 0 < \int_0^\infty g^q(x)\,dx < \infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)\,dx\right)^{1/p}\left(\int_0^\infty g^q(x)\,dx\right)^{1/q}, \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. We call (2) Hardy-Hilbert's integral inequality, which is important in analysis and its application (see[4]).
Yang [5,6] has extended inequality (2) by proving the following

If $f, g \ge 0,\ p > 1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda > 2-\min\{p,q\},$ such that

$$0<\int_0^\infty x^{1-\lambda} f^p(x)\,dx<\infty,\quad 0<\int_0^\infty x^{1-\lambda} g^q(x)\,dx<\infty\ ,$$

then the extended Hardy-Hilbert's inequality is given by

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dx\,dy < k_\lambda(p)\left(\int_0^\infty t^{1-\lambda} f^p(t)\,dt\right)^{1/p}\left(\int_0^\infty t^{1-\lambda} g^q(t)\,dt\right)^{1/q}, \tag{3}$$

where the constant $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right),$ is the best possible, B is the beta function.

Another new type of Hilbert's inequality is given by [7] as follows

Suppose $f(x), g(x) \ge 0,\ 0<\int_0^\infty f^2(x)\,dx<\infty,\ 0<\int_0^\infty g^2(x)\,dx<\infty$. Then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}}\,dx\,dy < c\left\{\int_0^\infty f^2(x)\,dx\int_0^\infty g^2(x)\,dx\right\}^{1/2}, \tag{4}$$

where $c=\sqrt{2}\left(\pi-2\tan^{-1}\sqrt{2}\right)=1.7408...$ is the best possible .

2. New Results

We state and prove the following :

Theorem 1.
Let $f(x), g(x), F(x), G(x) > 0,\ f(0)=g(0)=0,\ f(\infty)=g(\infty)=\infty, p>1,\ \frac{1}{p}+\frac{1}{q}=1,$ $\frac{1}{p}+\frac{1}{q}=1.$ *Then*

(i) *For* $\lambda > 1,$

$$\int_0^\infty\int_0^\infty \frac{F(x)G(y)}{\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)}\,dx\,dy \le C\left(\int_0^\infty \frac{f^{1-\lambda}(x)F^p(x)}{\left(f'(x)\right)^{p-1}}\,dx\right)^{1/p} \times\left(\int_0^\infty \frac{g^{1-\lambda}(y)G^q(y)}{\left(g'(y)\right)^{q-1}}\,dy\right)^{1/q}, \tag{5}$$

where $C = \frac{1}{\lambda-1}\left(\frac{1}{2^{\lambda-1}} - \frac{1}{2(3^{\lambda-1})}\right)$, *provtded the* int*egrals onthe right do sxists*.

(ii) *For* $\lambda = 1$,

(6) $$\int_0^\infty\int_0^\infty \frac{F(x)G(y)}{\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)}dx\,dy \le K\left(\int_0^\infty \frac{f^{\frac{p}{2}-1}(x)F^p(x)}{\left(f'(x)\right)^{p-1}}dx\right)^{1/p}$$
$$\times\left(\int_0^\infty \frac{g^{\frac{q}{2}-1}(y)G^q(y)}{\left(g'(y)\right)^{q-1}}dy\right)^{1/q},$$

where $K = \sqrt{2}\left(\pi - 2\tan^{-1}\sqrt{2}\right)$, *provided the* int*egrals on the right do exists*.

(iii) *For* $0 < \lambda < 1$ (*in fact* $0 < \lambda < \infty$),

(7) $$\int_0^\infty\int_0^\infty \frac{F(x)G(y)}{\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)^\lambda}dx\,dy \le A\left(\int_0^\infty \frac{f^{\left(1-\frac{\lambda}{2}\right)p-1}(x)F^p(x)}{\left(f'(x)\right)^{p-1}}dx\right)^{1/p}$$
$$\times\left(\int_0^\infty \frac{g^{\left(1-\frac{\lambda}{2}\right)q-1}(y)G^q(y)}{\left(g'(y)\right)^{q-1}}dy\right)^{1/q},$$

where

$A = \frac{1}{\sqrt{2}}\left(B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) - \int_{1/2}^{2}\frac{z^{\frac{\lambda}{2}-1}}{(1+z)^\lambda}dz\right)$, *provtded the* int*egrals on the right do exists.*

Proof of (i). We have

$$\int_0^\infty\int_0^\infty \frac{F(x)G(y)}{\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)^\lambda}dx\,dy$$
$$= \int_0^\infty\int_0^\infty \frac{F(x)\left(g'(y)\right)^{1/p}}{\left(f'(x)\right)^{1/q}\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)^{\lambda/p}}$$

$$\times\frac{G(y)\left(f'(x)\right)^{1/q}}{\left(g'(y)\right)^{1/p}\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)^{\lambda/q}}\,dx\,dy$$

$$\le \left(\int_0^\infty \int_0^\infty \frac{F^p(x)\, g'(y)}{\left(f'(x)\right)^{p/q} \left(f(x) + g(y) + \max\{f(x), g(y)\}\right)^\lambda} dx\, dy \right)^{1^{1/p}}$$

$$\times \left(\int_0^\infty \int_0^\infty \frac{G^q(y)\, f'(x)}{\left(g'(y)\right)^{q/p} \left(f(x) + g(y) + \max\{f(x), g(y)\}\right)^\lambda} dx\, dy \right)^{1/q}$$

$$= M^{1/p} N^{1/q}.$$

$$M = \int_0^\infty \frac{F^p(x)}{\left(f'(x)\right)^{p/q}} dx \int_0^\infty \frac{g'(y)}{\left(f(x) + g(y) + \max\{f(x), g(y)\}\right)^\lambda} dy\,.$$

Now

$$\int_0^\infty \frac{g'(y)}{\left(f(x) + g(y) + \max\{f(x), g(y)\}\right)^\lambda} dy$$

$$= \int_0^\infty \frac{1}{\left(f(x) + v + \max\{f(x), v\}\right)^\lambda} dv$$

$$= \int_0^{f(x)} \frac{1}{\left(2 f(x) + v\right)^\lambda} dv + \int_{f(x)}^\infty \frac{1}{\left(f(x) + 2v\right)^\lambda} \mathrm{dv}$$

$$= \frac{1}{\lambda - 1} \left(\frac{1}{2^{\lambda-1}} - \frac{1}{2\left(3^{\lambda-1}\right)} \right) f^{1-\lambda}(x) = C\, f^{1-\lambda}(x).$$

Therefore, we have

$$M = C \int_0^\infty \frac{f^{1-\lambda}(x) F^p(x)}{\left(f'(x)\right)^{p-1}} dx.$$

Similarly,

$$N = C \int_0^\infty \frac{g^{1-\lambda}(y) G^q(y)}{\left(g'(y)\right)^{q-1}} dy.$$

Proof of (ii).

$$\int_0^\infty \int_0^\infty \frac{F(x) G(y)}{\left(f(x) + g(y) + \max\{f(x), g(y)\}\right)^\lambda} dx\, dy$$

$$= \int_0^\infty\int_0^\infty \frac{F(x)\left(\frac{f^{1/2}(x)}{f'(x)}\right)^{1/q}\left(\frac{g'(y)}{g^{1/2}(y)}\right)^{1/p}}{\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)^{1/p}} \times \frac{G(y)\left(\frac{f'(x)}{f^{1/2}(x)}\right)^{1/q}\left(\frac{g^{1/2}(y)}{g'(y)}\right)^{1/p}}{\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)^{\lambda/q}}\,dx\,dy$$

$$\leq \left(\int_0^\infty\int_0^\infty \frac{F^p(x)\, f^{p/2q}(x)\, g'(y)}{\left(f'(x)\right)^{p/q} g^{1/2}(y)\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)}\,dy\right)^{1/p}$$

$$\times\left(\int_0^\infty\int_0^\infty \frac{G^q(y)\, g^{q/2p}(y)\, f'(x)}{\left(g'(y)\right)^{q/p} f^{1/2}(x)\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)}\,dx\,dy\right)^{1/q}$$

$$= P^{1/p} Q^{1/q}.$$

Considering

$$P = \int_0^\infty \frac{F^p(x)\, f^{p/2q}(x)}{\left(f'(x)\right)^{p/q}}\,dx \int_0^\infty \frac{g^{-1/2}(y)\, g'(y)}{f(x)+g(y)+\max\{f(x),g(y)\}}\,dy\,.$$

Now, we have

$$\int_0^\infty \frac{g^{-1/2}(y)\, g'(y)}{f(x)+g(y)+\max\{f(x),g(y)\}}\,dy = \int_0^\infty \frac{v^{-1/2}}{f(x)+v+\max\{f(x),v\}}\,dv$$

$$= \int_0^{f(x)} \frac{v^{-1/2}}{2f(x)+v}\,dv + \int_{f(x)}^\infty \frac{v^{-1/2}}{f(x)+2v}\,dv$$

$$= \sqrt{\frac{2}{f(x)}}\,\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + \sqrt{\frac{2}{f(x)}}\left(\frac{\pi}{2} - \tan^{-1} 2\right)$$

$$= \sqrt{\frac{2}{f(x)}}\left(\frac{\pi}{2} + \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \tan^{-1} 2\right)\,.$$

$$= \sqrt{\frac{2}{f(x)}}\left(\pi - \tan^{-1}\sqrt{2}\right) = \frac{K}{\sqrt{f(x)}}.$$

Therefore

$$.\qquad P = K\int_0^\infty \frac{f^{\frac{1}{2}\left(\frac{p}{q}-1\right)}(x)\, F^p(x)}{\left(f'(x)\right)^{p/q}}\,dx = K\int_0^\infty \frac{f^{\frac{p}{2}-1}(x)\, F^p(x)}{\left(f'(x)\right)^{p-1}}\,dx$$

Similarly,

$$Q = K\int_0^\infty \frac{g^{\frac{q}{2}-1}(y)\, G^q(y)}{\left(g'(y)\right)^{q-1}}\,dy\,.$$

Proof of (iii).

$$\int_0^\infty\int_0^\infty \frac{F(x)G(y)}{\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)^\lambda}\,dx\,dy$$

$$=\int_0^\infty\int_0^\infty \frac{F(x)\, g^{\left(\frac{\lambda}{2}-1\right)\frac{1}{p}}(y)\left(g'(y)\right)^{\frac{1}{p}}}{f^{\left(\frac{\lambda}{2}-1\right)\frac{1}{q}}(x)\left(f'(x)\right)^{\frac{1}{q}}\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)^{\frac{\lambda}{p}}}\,dy$$

$$\times\frac{G(y)\, f^{\left(\frac{\lambda}{2}-1\right)\frac{1}{q}}(x)\left(f'(x)\right)^{\frac{1}{q}}}{g^{\left(\frac{\lambda}{2}-1\right)\frac{1}{p}}(y)\left(g'(y)\right)^{\frac{1}{p}}\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)^{\frac{\lambda}{q}}}\,dxdy$$

$$\le\left(\int_0^\infty\int_0^\infty \frac{F^p(x)\, g^{\frac{\lambda}{2}-1}(y)\, g'(y)}{f^{\left(\frac{\lambda}{2}-1\right)\frac{p}{q}}(x)\left(f'(x)\right)^{\frac{p}{q}}\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)^{\lambda}}\,dx\,dy\right)^{1/p}$$

$$\times\left(\int_0^\infty\int_0^\infty \frac{G^q(y)\, f^{\frac{\lambda}{2}-1}(x)\, f'(x)}{g^{\left(\frac{\lambda}{2}-1\right)\frac{q}{p}}(y)\left(g'(y)\right)^{\frac{q}{p}}\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)^{\lambda}}\,dx\,y\right)^{1/q}$$

$$= R^{1/p}S^{1/q}.$$

We have

$$R=\int_0^\infty \frac{F^p(x)}{f^{\left(\frac{\lambda}{2}-1\right)\frac{p}{q}}(x)\left(f'(x)\right)^{\frac{p}{q}}}\,dx \quad \int_0^\infty \frac{g^{\frac{\lambda}{2}-1}(y)\, g'(y)}{\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)^{\lambda}}\,dy\,.$$

We consider the right integral

$$\int_0^\infty \frac{g^{\frac{\lambda}{2}-1}(y)\, g'(y)}{\left(f(x)+g(y)+\max\{f(x),g(y)\}\right)^{\lambda}}\,dy=\int_0^\infty \frac{v^{\frac{\lambda}{2}-1}}{\left(f(x)+v+\max\{f(x),v\}\right)^{\lambda}}\,dv$$

$$=\int_0^{f(x)} \frac{v^{\frac{\lambda}{2}-1}}{\left(2f(x)+v\right)^{\lambda}}\,dv+\int_{f(x)}^\infty \frac{v^{\frac{\lambda}{2}-1}}{\left(f(x)+2v\right)^{\lambda}}\,dv\,.$$

Now

$$\int_0^{f(x)} \frac{v^{\frac{\lambda}{2}-1}}{\left(2f(x)+v\right)}\,dv = \frac{1}{\left(2f(x)\right)^{\lambda/2}}\int_0^{f(x)} \frac{v^{\frac{\lambda}{2}-1}}{\left(1+\frac{v}{2f(x)}\right)^{\lambda}}\,dv$$

$$=\frac{1}{\left(2f(x)\right)^{\lambda/2}}\int_0^{1/2} \frac{z^{\frac{\lambda}{2}-1}}{\left(1+z\right)^{\lambda}}\,dz\,.$$

Similarly

$$\int_{f(x)}^{\infty} \frac{v^{\frac{\lambda}{2}-1}}{\left(f(x)+2v\right)^{\lambda}}\,dv = \frac{1}{\left(2f(x)\right)^{\lambda/2}}\int_{2}^{\infty}\frac{z^{\frac{\lambda}{2}-1}}{(1+z)^{\lambda}}\,dz\ ,.$$

Combining, we obtain

$$\int_{0}^{\infty}\frac{v^{\frac{\lambda}{2}-1}}{\left(f(x)+v+\max\{f(x),g(y)\}\right)^{\lambda}}\,dy = \frac{1}{\left(2f(x)\right)^{\lambda}}\left(\int_{0}^{\infty}-\int_{1/2}^{2}\right)\frac{z^{\frac{\lambda}{2}-1}}{(1+z)^{\lambda}}\,dz$$

$$= \frac{1}{\left(2f(x)\right)^{\lambda/2}}\left(B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)-\int_{1/2}^{2}\frac{z^{\frac{\lambda}{2}-1}}{(1+z)^{\lambda}}\,dz\right)$$

$$= \frac{A}{f^{\lambda/2}(x)}.$$

Therefore, we have

$$R = A\int_{0}^{\infty}\frac{f^{\left(1-\frac{\lambda}{2}\right)p-1}(x)F^{p}(x)}{\left(f'(x)\right)^{p-1}}\,dx$$

Similarly,

$$S = A\int_{0}^{\infty}\frac{g^{\left(1-\frac{\lambda}{2}\right)q-1}G^{q}(y)}{\left(g'(y)\right)^{q-1}}\,dy\ \ .$$

This completes the proof of the Theorem .

Remark. It may be mentioned that inequality (4) follows from Theorem 1 (ii), by putting

$$f(x)=g(x)=x,\quad p=q=2.$$

Theorem 2. *Let* $f(x), g(x), F(x), G(x) > 0,\ f(0)=g(0)=0,\ f(\infty)=g(\infty)=\infty,$

$p>1,\ \frac{1}{p}+\frac{1}{q}=1.$ *Then*

(i) *For* $\lambda > 1,$

(8) $$\int_{0}^{\infty}\int_{0}^{\infty}\frac{F(x)G(y)}{\left(f(x)+g(y)+\min\{f(x),g(y)\}\right)^{\lambda}}\,dx\,dy \le C\left(\int_{0}^{\infty}\frac{f^{1-\lambda}(x)F^{p}(x)}{\left(f'(x)\right)^{p-1}}\,dx\right)^{1/p}$$

$$\times\left(\int_0^\infty \frac{g^{1-\lambda}(y)G^q(y)}{(g'(y))^{q-1}}dy\right)^{1/q},$$

$where \quad C=\frac{1}{2(\lambda-1)}\left(1+3^{1-\lambda}\right),\ provided\ the\ \text{int}egrals\ on\ the\ gight\ do\ exists.$

(ii) $For\ \lambda=1,$

(9) $$\int_0^\infty\int_0^\infty \frac{F(x)G(y)}{\left(f(x)+g(y)+\min\{f(x),g(y)\}\right)^\lambda}dx\,dy \le K\left(\int_0^\infty \frac{f^{\frac{p}{2}-1}(x)F^p(x)}{(f'(x))^{p-1}}dx\right)^{1/p}$$

$$\times\left(\int_0^\infty \frac{g^{\frac{q}{2}-1}(y)G^q(y)}{(g'(y))^{q-1}}dy\right)^{1/q},$$

$where \quad K=2\sqrt{2}\tan^{-1}\sqrt{2},\ provided\ the\ \text{int}egrals\ on the\ right\ do exists.$

(iii) $For\ \ 0<\lambda<1\ \ (in\ \ fact\ \ 0<\lambda<\infty)),$

(10) $$\int_0^\infty\int_0^\infty \frac{F(x)G(y)}{\left(f(x)+g(y)+\min\{f(x),g(y)\}\right)^\lambda}dx\,dy \le A\left(\int_0^\infty \frac{f^{\left(1-\frac{\lambda}{2}\right)p-1}(x)F^p(x)}{(f'(x))^{p-1}}dx\right)^{1/p}$$

$$\times\left(\int_0^\infty \frac{g^{\left(1-\frac{\lambda}{2}\right)q-1}(y)G^q(y)}{(g'(y))^{q-1}}dy\right)^{1/q},$$

$where \quad A=B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)+\int_{1/2}^{2}\frac{z^{\frac{\lambda}{2}-1}}{(1+z)^\lambda}dz\ ,\ provided\ the\ \text{int}egrals\ on\ the\ right\ do\ exsits.$

Proof. It is quite similar to that given in Theorem 1, and therefore we omit it .

Corolary 3. $Let\ f(x),g(x)\ge 0,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda>0\ .Then$

(11) $$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda+\min\{x^\lambda,y^\lambda\}}dx\,dy \le \frac{K}{\lambda}\left(\int_0^\infty x^{\lambda\left(\frac{p}{2}-1\right)}f^p(x)\,dx\right)^{1/p}\left(\int_0^\infty x^{\lambda\left(\frac{q}{2}-1\right)}g^q(x)\,dx\right)^{1/q},$$

$where\quad K\ is\ as\ defined\ in\ theorem 1,\ \ provided\ the\ \text{int}egrals\ on\ the\ right\ do\ exists.$
$In\ particular\ \ fpr\ \lambda=1,\ \ p=q=2,\ \ we\ obtain\ (4).$

Proof. Follows from theorem 2, by putting f(x) = g(x) = x^λ .

Corollary 4. *Let* $f(x), g(x) \geq 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ \lambda > 0$. *Then*

(12) $$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda + \min\{x^\lambda, y^\lambda\}} dx\,dy \leq \frac{K}{\lambda}\left(\int_0^\infty x^{\lambda\left(\frac{p}{2}-1\right)} f^p(x)\,dx\right)^{1/p}\left(\int_0^\infty x^{\lambda\left(\frac{q}{2}-1\right)} g^q(x)\,dx\right)^{1/q},$$

whereKis as defined in theorem 2, *provided the* int*egrals on the right do exists.*

In particular for $\lambda = 1,\ p = q = 2,$ *we have*

(13) $$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x + y + \min\{x, y\}} dx\,dy \leq K\left(\int_0^\infty f^2(x)\,dx\right)^{1/2}\left(\int_0^\infty g^2(x)\,dx\right)^{1/2}.$$

References

[1] G.H.Hardy, Note on a theorem of Hilbert concerning series if positive terms, Proc. Math. Soc., 23(2) (1925), Records of Proc. XLV-XLVI.

[2] G.H.Hardy, J.E.Littlewood and G.Polya, Inequalities,Cambridge University Press, Cambridge, UK, 1952.

[3] Xin Li and C.-P. Chen, Inequalities for the gamma function, J. Ineq. Pure Appl. Math. Vol 8, Issue 4, Article 28, (2007) .

[4] D.S.Mitrinovic, J.E.Pecaric and A.M.Fink, Inequalities Involving Functions and their Integrals and Derivatives, Kluwer Academic Publisher, Bosten, 1991.

[5] B. Yang, On a generalization of Hardy-Hilbert's integrel inequality with a best value Chinese Ann. Math. Ser A 21 (2000) 401-408.

[6] B. yang, On Hardy-Hilbert's integral inequality, J Math. Anal. Appl. 26 (2001) 295-306.

[7] Yongjin Li, Jing Wu, and Bing He, A new Hilbert –type inequality and the Equivalent form, Int. J. Math. Math. Sci., 3006 (3006), Article ID 45378,Page 1-6.

Received: April 1, 2008