International Mathematical Forum, 3, 2008, no. 43, 2129 - 2137

On $|R, p_n|_k$ Summability of Infinite Series

W. T. Sulaiman

Dept. of Mathematics, College of Computer Sciences
University of Mosul, Mosul, Iraq
waadsulaiman@hotmail.com

Abstract

New theorem concerning $|R, p_n|_k$ summability of an infinite series has been proved of an infinite series has been proved. Some other results are also deduced.

Keywords: Absolute summability, weight function

1. Introduction

Let $\sum a_n$ be a given infinite series with sequence of partial sums (s_n). By u_n we denote the nth (C,1) mean of the sequence (s_n). The series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if

(1) $$\sum_{n=1}^{\infty} n^{k-1}|u_n - u_{n-1}|^k < \infty .$$

Let (p_n) be a sequence of positive numbers such that

(2) $$P_n = \sum_{v=0}^{n} p_v \to \infty, \quad as \ n \to \infty \qquad (P_{-1} = p_{-1} = 0).$$

The sequence -to – sequence transformation

(3) $$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (t_n) of the Riesz means of the sequence (s_n), generated by the sequence of coefficients (p_n). We say that $\sum a_n$ is summable $|R, p_n|_k$, $k \geq 1$, if

(4) $$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty ,$$

and it is said to be summable $|R, p_n, \delta|_k$, $k \geq 1$, $\delta \geq 0$, if

(5) $$\sum_{n=1}^{\infty} n^{\delta k+k-1} |t_n - t_{n-1}|^k < \infty .$$

In the special case when $p_n = 1$ for all values of n (resp. k =1), $|R, p_n|_k$ summability is the same as $|C,1|_k$ $(resp. |R, p_n|)$ summability . A series $\sum a_n$ is said to be summable $|\overline{N}, p_n, \delta|_k$, $k \geq 1$, $\delta \geq 0$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k+k-1} |t_n - t_{n-1}|^k < \infty .$$

A positive sequence (α_n) is said to be quasi β – power increasing sequence if there exists a constant $K = K(\alpha, \beta) \geq 1$ such that

$$K\, n^{\beta} \alpha_n \geq m^{\beta} \alpha_m$$

Holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is quasi β-power increasing sequence for any $\beta > 0$, but the converse need not be true as can be seen by taking $\alpha_n = n^{-\beta}$, $\beta > 0$.

Bor [1] proved the following

Theorem . Let (X_n) be a positive nondecreasing sequence and the sequences (λ_n) and (β_n) be such that

(6) $$|\Delta \lambda_n| \leq \beta_n ,$$

(7) $$\beta_n \to 0 \quad as \quad n \to \infty ,$$

(8) $$\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty .$$

(9) $$X_n |\lambda_n| = O(1).$$

If (p_n) is a sequence satisfying

(10) $$P_n = O(np_n),$$

(11) $$\sum_{n=v+1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}} = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{P_v}\right),$$

(12) $$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|r_n|^k = O(X_m) \ as \ \ m\to\infty,$$

where

$$r_n = \frac{1}{n+1}\sum_{v=1}^{n} v\,a_v ,$$

then the series $\sum a_n\lambda_n$ is summable $|N, p_n, \delta|_k$, $k\geq 1,\ 0<\delta<1/k$.

The object of this paper is to give the following

2. New Result

Theorem . Let $(X_n),(\lambda_n),(\phi_n)$ be sequences such that, (X_n) is a quasi β-power Increasing sequence for some $0<\beta<1$, and such that they are satisfying (9),(10), (16), (17)

In addition to the following

(13) $$\sum_{n=v+1}^{\infty} n^{\delta k+k-1}\frac{p_n^k}{P_n^k P_{n-1}} = O\left(v^{\delta k+k-1}\frac{p_v^{k-1}}{P_v^k}\right),$$

(14) $$\sum_{n=1}^{m}\frac{n^{\delta k+k-1}p_n^k|s_n|^k|\phi_n|^k}{P_n^k X_n^{k-1}} = O(X_m),$$

(15) $$\sum_{n=1}^{m}\frac{n^{\delta k+k-1}|s_n|^k|\Delta\phi_n|^k}{X_n^{k-1}} = O(X_m).$$

Then the series $\sum a_n\lambda_n\phi_n$ is summable $|R, p_n, \delta|_k$, $k\geq 1,\ \delta\geq 0$.

Remark. In comparing with (12), dividing by X_n^{k-1}, $k>1$, in (14) and (15) leeds to gain powers , and hence weaking the conditions .

3. Lemma

The following lemma is needed

Lemma. If (X_n) is a quasi β-power increasing sequence for some $0<\beta<1$, then under the conditions

(16) $$\lambda_n \to 0 \ \ as \ n \to \infty,$$

(17) $$\sum_{n=1}^{\infty} nX_n \left|\Delta\left|\Delta\lambda_n\right|\right| < \infty,$$

we have

(18) $$nX_n\left|\Delta\lambda_n\right| = O(1) \ as \ n \to \infty,$$

(19) $$\sum_{n=1}^{\infty} X_n\left|\Delta\lambda_n\right| < \infty.$$

Proof. Since $\lambda_n \to 0$, then $\Delta\lambda_n \to 0$, and hence

$$\begin{aligned}
n\,X_n\left|\Delta\lambda_n\right| &= n\,X_n\left|\sum_{v=n}^{\infty}\Delta\left|\Delta\lambda_n\right|\right| \\
&= O(1)\,nX_n\sum_{v=n}^{\infty}\left|\Delta\left|\Delta\lambda_n\right|\right| \\
&= O(1)\; n^{1-\beta}\left(n^{\beta}X_n\right)\sum_{v=n}^{\infty}\left|\Delta\left|\Delta\lambda_v\right|\right| \\
&= O(1)\,n^{1-\beta}\sum_{v=n}^{\infty}v^{\beta}X_v\left|\Delta\left|\Delta\lambda_v\right|\right| \\
&= O(1)\sum_{v=n}^{\infty}v^{1-\beta}v^{\beta}X_v\left|\Delta\left|\Delta\lambda_v\right|\right| \\
&= O(1)\sum_{v=n}^{\infty}vX_v\left|\Delta\left|\Delta\lambda_v\right|\right| = O(1).
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty}X_n\left|\Delta\lambda_n\right| &= \sum_{n=1}^{\infty}X_n\sum_{v=n}^{\infty}\Delta\left|\Delta\lambda_v\right| \\
&= O(1)\sum_{v=1}^{\infty}\left|\Delta\left|\Delta\lambda_v\right|\right|\sum_{n=1}^{v}X_n \\
&= O(1)\sum_{v=1}^{\infty}\left|\Delta\left|\Delta\lambda_v\right|\right|\sum_{n=1}^{v}\left(n^{\beta}X_n\right)n^{-\beta} \\
&= O(1)\sum_{v=1}^{\infty}\left|\Delta\left|\Delta\lambda_v\right|\right|v^{\beta}X_v\sum_{n=1}^{v}n^{-\beta} \\
&= O(1)\sum_{v=1}^{\infty}v^{\beta}X_v\left|\Delta\left|\Delta\lambda_v\right|\right|\int_0^{v}x^{-\beta}dx
\end{aligned}$$

$$= O(1)\sum_{v=1}^{\infty} v^{\beta} X_v |\Delta|\Delta\lambda_v|| v^{1-\beta}$$
$$= O(1)\sum_{v=1}^{\infty} vX_v |\Delta|\Delta\lambda_v|| = O(1).$$

4. Proof of theorem 1

Let (T_n) be the sequence of (R, p_n) means of the series $\sum a_n\phi_n\lambda_n$. Then by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r\phi_r\lambda_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v+1})a_v\phi_v\lambda_v .$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}\, a_v\, \phi_v\, \lambda_v$$
$$= \frac{p_n}{P_nP_{n-1}}\left(\sum_{v=1}^{n-1}\left(\sum_{r=1}^{v} a_v\right)\Delta(P_{v-1}\phi_v\lambda_v) + \left(\sum_{v=1}^{n} a_v\right)P_{n-1}\phi_n\lambda_n\right)$$
$$= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\left(-\, p_v\, s_v\phi_v\lambda_v + P_vs_v\Delta\lambda_v\phi_v + P_vs_v\lambda_{v+1}\Delta\phi_v\right) + \frac{p_n}{P_n}s_n\phi_n\lambda_n$$
$$= T_1 + T_2 + T_3 + T_4 .$$

In order to prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|T_j|^k < \infty, \qquad j = 1,2,3,4.$$

Applying Holder's inequality, we have

$$\sum_{n=2}^{m+1} n^{\delta k+k-1}|T_1|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} p_vs_v\phi_v\lambda_v\right|^k$$
$$\leq \sum_{n=2}^{m+1} n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v|s_v|^k|\phi_v|^k|\lambda_v|^k \left(\sum_{v=1}^{n-1}\frac{p_v}{P_{n-1}}\right)^{k-1}$$
$$= O(1)\sum_{v=1}^{m} p_v|s_v|^k|\phi_v|^k|\lambda_v|^k \sum_{n=v+1}^{m+1} n^{\delta k+k-1}\frac{p_n^k}{P_n^kP_{n-1}}$$

$$= O(1)\sum_{v=1}^{m} v^{\delta k+k-1}\frac{p_v^k}{P_v^k}|s_v|^k|\phi_v|^k|\lambda_v|^k$$

$$= O(1)\sum_{v=1}^{m}\frac{v^{\delta k+k-1}p_v^k|s_v|^k|\phi_v|^k}{P_v^k X_v^{k-1}}(X_v|\lambda_v|)^{k-1}|\lambda_v|$$

$$= O(1)\sum_{v=1}^{m}\frac{v^{\delta k+k-1}p_v^k|s_v|^k|\phi_v|^k}{P_v^k X_v^{k-1}}|\lambda_v|$$

$$= O(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\frac{r^{\delta k+k-1}p_r^k|s_r|^k|\phi_r|^k}{P_r^k X_r^{k-1}}\right)\Delta|\lambda_v| + O(1)\left(\sum_{v=1}^{m}\frac{v^{\delta k+k-1}p_v^k|s_v|^k|\phi_v|^k}{P_v^k X_v^{k-1}}\right)|\lambda_m|$$

$$= O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_v| + O(1)X|\lambda_m| = O(1).$$

$$\sum_{n=2}^{m+1} n^{\delta k+k-1}|T_2|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1}\left|\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1}P_v s_v\Delta\lambda_v\phi_v\right|^k$$

$$\le \sum_{n=2}^{m+1} n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}^k}\sum_{v=1}^{n-1}\frac{P_v^k|s_v|^k|\phi_v|^k}{X_v^{k-1}}|\Delta\lambda_v|\left(\sum_{v=1}^{n-1}X_v|\Delta\lambda_v|\right)^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\frac{P_v^k|s_v|^k|\phi_v|^k}{X_v^{k-1}}|\Delta\lambda_v|\sum_{n=v+1}^{m+1} n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}^k}$$

$$= O(1)\sum_{v=1}^{m}\frac{P_v|s_v|^k|\phi_v|^k}{X_v^{k-1}}|\Delta\lambda_v|\sum_{n=v+1}^{m+1} n^{\delta k+k-1}\frac{p_n^k}{P_n^k P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\frac{v^{\delta k+k-1}p_v^{k-1}|s_v|^k|\phi_v|^k}{P_v^{k-1}X_v^{k-1}}|\Delta\lambda_v|$$

$$= O(1)\sum_{v=1}^{m}\frac{v^{\delta k+k-1}p_v^k|s_v|^k|\phi_v|^k}{P_v^k X_v^{k-1}}(v|\Delta\lambda_v|) \qquad \text{(by(10))}$$

$$= O(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\frac{r^{\delta k+k-1}p_r^k|s_r|^k|\phi_r|^k}{P_r^k X_r^{k-1}}\right)|\Delta|\Delta\lambda_v|| + O(1)\left(\sum_{v=1}^{m}\frac{v^{\delta k+k-1}p_v^k|s_v|^k|\phi_v|^k}{P_v^k X_v^{k-1}}\right)m|\Delta\lambda_m|$$

$$= O(1)\sum_{v=1}^{m-1}X_v|\Delta(v\Delta\lambda_v)| + O(1)\,mX_m|\Delta\lambda_m|$$

$$= O(1)\sum_{v=1}^{m-1} X_v\left|\Delta\lambda_v\right| + O(1)\sum_{v=1}^{m-1} vX_v\left|\Delta\left|\Delta\lambda_v\right|\right| + O(1)\,mX_m\left|\Delta\lambda_m\right|$$
$$= O(1).$$

$$\sum_{n=2}^{m+1} n^{\delta k+k-1}\left|T_3\right|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1}\left|\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} P_v s_v \Delta\phi_v \lambda_{v+1}\right|^k$$
$$\le \sum_{n=2}^{m+1} n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\frac{P_v^k}{p_v^{k-1}}\left|s_v\right|^k\left|\Delta\phi_v\right|^k\left|\lambda_{v+1}\right|^k\left(\sum_{v=1}^{n-1}\frac{p_v}{P_{n-1}}\right)^{k-1}$$
$$= O(1)\sum_{v=1}^{m}\frac{P_v^k}{p_v^{k-1}}\left|s_v\right|^k\left|\Delta\phi_v\right|^k\left|\lambda_{v+1}\right|^k\sum_{n=v+1}^{m+1} n^{\delta k+k-1}\frac{p_n^k}{P_n^k P_{n-1}}$$
$$= O(1)\sum_{v=1}^{m} v^{\delta k+k-1}\left|s_v\right|^k\left|\Delta\phi_v\right|^k\left|\lambda_{v+1}\right|^k$$
$$= O(1)\sum_{v=1}^{m}\frac{v^{\delta k+k-1}\left|s_v\right|^k\left|\Delta\phi_v\right|^k}{X_{v+1}^{k-1}}\left(X_{v+1}\left|\lambda_{v+1}\right|\right)^{k-1}\left|\lambda_{v+1}\right|$$
$$= O(1)\sum_{v=1}^{m}\frac{v^{\delta k+k-1}\left|s_v\right|^k\left|\Delta\phi_v\right|^k}{X_v^{k-1}}\left|\lambda_{v+1}\right|$$

$$= O(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\frac{r^{\delta k+k-1}\left|s_r\right|^k\left|\Delta\phi_r\right|^k}{X_r^{k-1}}\right)\Delta\left|\lambda_{v+1}\right| +$$
$$O(1)\left(\sum_{v=1}^{m}\frac{v^{\delta k+k-1}\left|s_v\right|^k\left|\Delta\phi_v\right|^k}{X_v^{k-1}}\right)\left|\lambda_{m+1}\right|$$
$$= O(1)\sum_{v=1}^{m-1} X_v\left|\Delta\lambda_{v+1}\right| + O(1)\,X_m\left|\lambda_{m+1}\right|$$
$$= O(1)\sum_{v=1}^{m-1} X_{v+1}\left|\Delta\lambda_{v+1}\right| + O(1)\,X_{m+1}\left|\lambda_{m+1}\right|$$
$$= O(1).$$

$$\sum_{n=1}^{m} n^{\delta k+k-1}\left|T_4\right|^k = \sum_{n=1}^{m} n^{\delta k+k-1}\left|\frac{p_n}{P_n}s_n\phi_n\lambda_n\right|^k$$
$$= O(1)\sum_{v=1}^{m} n^{\delta k+k-1}\frac{p_n^k}{P_n^k}\left|s_n\right|^k\left|\phi_n\right|^k\left|\lambda_n\right|^k$$
$$= O(1),\ \text{ as in the case of } T_1 .$$

This completes the proof of the theorem .

5. Applications

Theorem 2. Let $(X_n),(\lambda_n)$ be sequences such that (X_n) is a quasi $\beta-$power increasing sequence for some $0<\beta<1,$ and they are satisfying conditions (9),(10), (13), (16), (17) in addition

(20)
$$\sum_{n=1}^{m}\frac{n^{\delta k+k-1}p_n^k|s_n|^k}{P_n^k X_n^{k-1}}=O(X_m).$$

Then the series $\sum a_n\lambda_n\phi_n$ is summable $|R,p_n,\delta|_k$, $k\geq 1,$ $\delta\geq 0$.

Proof. Follows from theorem 1, by putting $\phi_n=1,$ for all n .

Corollary 3. Let $(X_n),(\lambda_n)$ be sequences such that (X_n) is a quasi $\beta-$power increasing sequence for some $0<\beta<1,$ and such that they are satisfying conditions (9), (16), (17), and

(21)
$$\sum_{n=1}^{\infty}\frac{n^{\delta k-1}|s_n|^k}{X_n^{k-1}}=O(X_m).$$

Then the series $\sum a_n\lambda_n$ is summable $|C,1,\delta|_k$, $k\geq 1,$ $0\leq\delta<1/k$.

Proof. This follows from theorem 2, by putting $p_n=1,$ for all n. The condition (13) Is obviously satisfied as for $p_n=1,$

$$\sum_{n=v+1}^{\infty}\frac{b^{\delta k+k-1}}{n-1}=O(1)\sum_{n=v+1}^{\infty}n^{\delta k+k-2}=O(1)\int_v^{\infty}x^{\delta k+k-2}dx=O\left(v^{\delta k+k-1}\right).$$

Theorem 4. Let $(X_n),(\lambda_n)$ be sequences such that (X_n) is a quasi $\beta-$power increasing sequence for some $0<\beta<1,$ and such that they are satisfying conditions (9), (10),(13), (16), (17), (21), in addition

(22)
$$p_{n+1}=O(p_n),$$

(23)
$$n|\Delta p_n|=O(p_n).$$

Then the series $\sum\frac{a_n\lambda_n P_n}{np_n}$ is summable $|R,p_n,\delta|_k$, $k\geq 1,$ $\delta\geq 0$.

Proof. The proof follows from theorem 1, by putting $\phi_n = \frac{P_n}{np_n}$. Just it needs to show that $\left|\Delta\left(\frac{P_n}{np_n}\right)\right| = O\left(\frac{1}{n}\right)$, under the conditions (22) and (23) . This can be shown as follows

$$\Delta\left(\frac{P_n}{np_n}\right) = \frac{P_n}{np_n} - \frac{P_{n+1}}{(n+1)p_{n+1}} \le P_{n+1}\left(\frac{(n+1)p_{n+1} - np_n}{n(n+1)p_n p_{n+1}}\right)$$

$$= P_{n+1}\frac{\left(p_{n+1} - n\Delta p_n\right)}{n(n+1)p_n p_{n+1}} = O\left(\frac{p_{n+1} + n|\Delta p_n|}{np_n}\right) \quad \text{(by(10))}$$

$$= O\left(\frac{p_{n+1}}{np_n} + O\left(\frac{1}{n}\right)\right) = O\left(\frac{1}{n}\right).$$

References

[1] H. Bor, On $\left|\overline{N}, p_n, \delta\right|_k$ summability factors of infinite series, Taiwanese Journal of Mathematics, 1 (1997) 327-332.

Received: April 1, 2008