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On the Growth of Semiconjugated Entire Functions

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Abstract. Let f and h be two entire functions and let $g : \mathbb{C} \to \mathbb{C}$ be a non-constant continuous function such that $f \circ g = g \circ h$. Then we say that f and h are semiconjugated by g and we call g a semiconjugacy. In this paper, we have shown that under some conditions on the growth of semiconjugacy, the growth of semiconjugated entire functions can be controlled. We also have proved a result on asymptotic values of derivative of composition of entire functions.

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1. Introduction

Let f and h be two entire functions and let $g : \mathbb{C} \to \mathbb{C}$ be a non-constant continuous function such that

$$f \circ g = g \circ h, \tag{1.1}$$

It is easy to see that there are countably many entire functions which satisfy (1.1). In (1.1), we say that f and h are semiconjugated by g and we call g a semiconjugacy. Further, if $f = h$, then (1.1) reduces to $h \circ g = g \circ h$ and in this case g and h becomes permutable entire functions. W. Bergweiler and A. Hinkkanen in [**2**] has shown that the non-constant continuous function need not be open or discrete even if g and h are permutable entire functions. We also recall that a point $a \in \mathbb{C}$ is said to be an asymptotic value (or transcendental singularity) of f if there exists a curve $\Gamma : [0, 1) \to \mathbb{C}$ such that $\lim_{t\to 1} \Gamma(t) = \infty$ and $\lim_{t\to 1}(f \circ \Gamma)(t) = a$. In this case we say that $a \in AV(f)$. The role of asymptotic values was utilized by Baker and Singh [**1**] to show that the entire function $e^{e^z} - e^z$ has no wandering domain. They utilized the properties that if f is an entire function of finite order K then f has atmost $2K$ asymptotic values ([**6**], page 307). They further showed that if f and g are entire functions having finite number of asymptotic values then so does $f \circ g$ [**1**]. Later Bergweiler and Wang [**3**] proved that if f and g are two entire functions, then

$$AV(f \circ g) \subset AV(f) \cup f(AV(g)).$$

Recently Singh and Sharma [**7**] obtained some results on asymptotic values of semiconjugated entire functions. In this paper we have shown that under some conditions on the growth of semiconjugacy, the growth of semiconjugated entire functions can be controlled. As a consequence of our main result we too have proved a result on asymptotic values of derivative of composition of entire functions.

We recall that an entire function f is of finite *order* if there is a positive constant α and $r_0 > 0$ such that $|f(z)| < \exp|z|^\alpha$ for $|z| > r_0$. If f is of finite order then the number

$$\rho_f = \inf\{\alpha : |f(z)| < \exp|z|^\alpha \text{ for } z \text{ sufficiently large }\}$$

is called the *order* of f. Further if

$$M(r, f) = \max\{|f(z)| : |z| \le r\},$$

then for given $\epsilon > 0$, there exists $r_0 > 0$ such that $M(r, f) < \exp r^{\rho+\epsilon}$ for $r > r_0$ and $M(r, f) > \exp r^{\rho-\epsilon}$ for a sequence of $r = r_n \to \infty$. Thus order of f is given by

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}$$

Likewise, we denote the lower order of f by λ_f and is defined as

$$\lambda_f = \liminf_{r\to\infty} \frac{\log\log M(r, f)}{\log r}$$

2. Growth of semiconjugated entire functions

Main result of this paper is

Theorem 1. *Let f, g and h be three entire functions such that $f \circ g = g \circ h$. Further, let ρ_f and ρ_g are finite. Then ρ_h is finite.*

For the proof of above Theorem we need the following lemmas.

The first one is due to J. Cluine.

Lemma 2. [**4**] Let f and g be entire functions with $g(0) = 0$. Let α satisfy $0 < \alpha < 1$ and let $c(\alpha) = (1-\alpha)^2/4\alpha$. Then for $r > 0$,

$$M(r, f \circ g) \ge M(c(\alpha)M(\alpha r, g), f).$$

Further, if g is any entire function, then with $\alpha = 1/2$, for sufficiently large values of r,

$$M(r, f \circ g) \ge M(\frac{1}{8}M(\frac{r}{2}, g) - |g(0)|, f).$$

Lemma 3. Let f and g be two entire functions. Then

$$M(r, f\circ g) \le M(M(r,g), f).$$

Proof. Let $D = \{z : |z| \le r\}$. Now $g(D) \subset (z : |z| \le M(r,g))$. Thus $f(g(D)) \subset f(|z| \le M(r,g))$ for every z such that $|z| \le r$, and so $\max|(f\circ g)(D)| \le M(M(r,g), f)$. That is, $M(r, f\circ g) \le M(M(r,g), f)$. This completes the proof.

Proof of the Theorem 1. Suppose $\rho_h = \infty$. Then given any $K > \rho_g + 1$, there is a sequence r_n such that $M(r_n, h) > \exp(r_n^K)$. Now since order of f and g are finite and so for $0 < \epsilon < 1$, there exists a positive number r_0 such that

$$M(r,f) < \exp(r)^{\rho_f+\epsilon} \quad \text{and} \quad M(r,g) < \exp(r)^{\rho_g+\epsilon} \text{ for every } r \ge r_0.$$

Thus by Lemma 3, we have

$$M(r, f\circ g) \le M(M(r,g), f) < M(\exp(r^{\rho_g+\epsilon}), f) < \exp(\exp(r^{\rho_g+\epsilon}))^{\rho_f+\epsilon}$$

for every $r \ge r_0$. Again by Lemma 2, we have

$$M(r, g\circ h) \ge M(\frac{1}{8}M(\frac{r}{2}, h) - |h(0)|, g) \ge M(\frac{1}{16}M(\frac{r}{2}, h), g)$$

for every $r \ge r_0$. Now lower order of g is finite so

$$M(r,g) > \exp(r)^{\lambda_g-\epsilon} \quad \text{for every} \quad r \ge r_0.$$

Thus for a sequence $r = r_n > r_0$, we have

$$M(r_n, g\circ h) \ge M(\frac{1}{16}\exp(r_n/2)^K, g) > \exp\left(\frac{1}{16}\exp(r_n/2)^K\right)^{\lambda_g-\epsilon}$$

and so

$$\exp(\exp({r_n}^{\rho_g+\epsilon})^{\rho_f+\epsilon} > \exp\left(\frac{1}{16}\exp(r_n/2)^K\right)^{\lambda_g-\epsilon}$$

which implies that

$$\exp({r_n}^{\rho_g+\epsilon})^{\rho_f+\epsilon} > \left(\frac{1}{16}\exp(r_n/2)^K\right)^{\lambda_g-\epsilon} = \left(\frac{1}{16}\right)^{\lambda_g-\epsilon}\exp((\lambda_g-\epsilon)(r_n/2)^K)$$

Thus we have

$$\rho_f + \epsilon > \frac{(\lambda_g-\epsilon)\log(1/16)}{r_n^{\rho_g+\epsilon}} + \frac{\lambda_g-\epsilon}{2^K} r_n^{K-(\rho_g+\epsilon)}.$$

Now first part on the right hand side approches zero as $r \to \infty$ and since $K > \rho_g + \epsilon$, the second term on the right hand side approaches ∞ as $r \to \infty$, a contradiction. Hence ρ_h is finite.

One may think that if f, g and h are three entire functions such that ρ_f and ρ_g are finite, and $f\circ g = g\circ h$, then either $\rho_h \le \rho_f$ or $\rho_h \le \rho_g$.
However, the following example illustrates that this is not the case.

Example 4. Let $f(z) = ze^{2z}$, $g(z) = z^2$ and $h(z) = ze^{z^2}$ be three entire functions. Then $f\circ g = g\circ h$ and ρ_h is greater than orders of both f and g. Again, following example illustrates that in Theorem 1, the order of h can be arbitarily large.

Example 5. Let $f(z) = e^z$, $g(z) = e^{z^n}$ and $h(z) = e^{z^n/n}$, $n \in \mathbb{N}$ be three entire functions in which $\rho_f = 1$, $\rho_g = \rho_h = n$ and $f \circ g = g \circ h$.

Theorem 6. *Let g and h be two entire functions such that $g \circ g = h \circ g$. Then $g = h$.*
Proof. Suppose not, then there exist atleast one point z_0 such that $g(z_0) \neq h(z_0)$. **Case (i).** Let there exists $\zeta \in \mathbb{C}$ such that $g(\zeta) = z_0$. Then $g(g(\zeta)) = g(z_0)$ and $h(g(\zeta)) = h(z_0)$, a contradiction to $g \circ g = h \circ g$.
Case (ii). Let there exist atleast two points say z_0 and z_1 such that $g(z_0) \neq h(z_0)$ and $g(z_1) \neq h(z_1)$, then by Picard's theorem atleast one of z_0 and z_1, say z_0 is not Picard exceptional value.Therefore, there exists $\zeta \in \mathbb{C}$ such that $g(\zeta) = z_0$. Thus as in the case (i) we get a contradiction.
Case (iii). Suppose z_0 is the only point and is such that z_0 is omitted by every $\zeta \in \mathbb{C}$ and $g(z_0) \neq h(z_0)$. Then there does not exists $\zeta \in \mathbb{C}$ such that $g(\zeta) = z_0$ and $g(z_0) \neq h(z_0)$. Since g is non-constant analytic and hence open so we can find $z_i \in \mathbb{C}$ $(i = 1, 2, 3, 4, 5)$ such that $g(z_i) \neq g(z_0)$. Let $g(z_i) = \alpha_i$. Then the set $g^{-1}(\alpha_i)$ does not contain z_0, for otherwise z_0 has two images $g(z_i) = \alpha_i$ and $g(z_0)$, which is not possible. Thus $g^{-1}(z_i) = h^{-1}(z_i)$ as z_0 is the only point with $g(z_0) \neq h(z_0)$. Thus g and h share five values and hence by Nevanlinna's uniqueness theorem [**5**]**,** $g \equiv h$, a contradiction to $g(z_0) \neq h(z_0)$. This completes the proof.
In the above theorem we observe that if $g \circ g = h \circ g$, then $g = h$. However, if $g \circ g = g \circ h$, then h need not be same as g. For example, consider the following: Let $g(z) = e^z$ and $h(z) = e^z + 2\pi i$. Clearly $g(z) \neq h(z)$. But

$$(g \circ g)(z) = e^{e^z} = (g \circ h)(z).$$

In fact, if $g(z)$ is periodic with period k, and $h(z) = g(z) + k$. then $(g \circ g)(z) = (g \circ h)(z)$. Also, there exist non-periodic entire functions g and h such that $g \circ g = g \circ h$. For, let $g(z) = z^2 e^{z^2}$ and $h(z) = -(z^2 e^{z^2})$. Then

$$g(h(z)) = (z^2 e^{z^2})^2 e^{(z^2 e^{z^2})^2} = g(g(z)).$$

As an application of Theorem 1, we show that if f, g and h are three entire functions satisfying (1.1) and f and g are of finite order, then derivative of $f \circ g$ have finite number of asymptotic values.
Theorem 7. *Let f, g and h be three entire functions such that $f \circ g = g \circ h$. Further, let ρ_f and ρ_g are finite. Then the derivative of $f \circ g$ have finite number of asymptotic values.*
In order to prove this result we need the following Lemma.
Lemma 8. Let F and G be entire functions having finite number of asymptotic values. Then $F \times G$ has finite number of asymptotic values.
Proof. Let $a_1, a_2, \cdots a_n$ and $b_1, b_2, \cdots b_m$ be asymptotic values of F and G respectively. Suppose $H = F \times G$ has infinite number of asymptotic values. Then we can choose an asymptotic value of H, say α such that $\alpha \neq a_i b_j$ for every $i = 1, 2, \cdots n$ and $j = 1, 2, \cdots m$. Now as α is an asymptotic value of H,

there exists a curve Γ such that $H(z) \to \alpha$ as $z \to \infty$ along Γ. As F and G are entire functions, $F(z)$ and $G(z)$ tends to some value say A and B finite or infinite as $z \to \infty$ along Γ. That is, A and B are asymptotic values of F and G respectively. Therefore, $A = a_i$ for some i and $B = b_j$ for some j. Thus $H(z) \to a_i b_j$ as $z \to \infty$ along Γ. Therefore, $\alpha = a_i b_j$, a contradiction. Hence H have finite number of asymptotic values.

Proof of Theorem 7. Since ρ_f and ρ_g are finite and so by Theorem 1, ρ_h is finite. Also $\rho_f = \rho_{f'}$, $\rho_g = \rho_{g'}$ and $\rho_h = \rho_{h'}$ are finite. Now f, g and h have finite number of asymptotic values which in turns implies that f', g' and h' also have finite number of asymptotic values. Therefore, $g'(h(z)) = F$ has finite number of asymptotic values and so by Lemma 8, $F \times G$, where $G = h'(z)$ have finite number of asymptotic values.

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