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Common Fixed Points of Increasing Operators in Posets and Related Semilattice Properties

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Abstract

An extension of a well known theorem of R. DeMarr [Common fixed points for isotone mappings, Colloquium Math., 13 (1964), 45-48] for commuting families of increasing operators in posets is studied and we establish two corollaries concerning semilattice properties of the set of their common fixed points.

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1. Preliminaries

Throughout this paper $X = (X, \leq)$ denotes a poset, $\mathcal{A}_X = \{A_\gamma: \gamma \in \Gamma, A_\gamma: X \rightarrow X\}$ stands for a commutative family of increasing operators of X and $\mathrm{Fix}(\mathcal{A}_X) = \{x \in X : x = A_\gamma(x) \text{ for every } \gamma \in \Gamma\}$ is the set of common fixed points of the operators of the family $\mathcal{A}_X$. DeMarr [3] proved the following well known theorem (cfr. also [7]):

Theorem 1. *Let* $\emptyset \neq X$ *be a poset and* $\mathcal{A}_X = \{A_\gamma: \gamma \in \Gamma, A_\gamma: X \rightarrow X\}$ *be a commutative family of increasing operators of* X. *Assume that*
(1) *there exists a point* $u \in X$ *such that* $u \leq A_\gamma(u)$ *for every* $\gamma \in \Gamma$,

(2) *any chain* C of X *containing* u *has supremum in* X,
then Fix($\mathcal{A}_X$) $\neq \varnothing$.

In [4] the first author proved the following result:

Theorem 2. *Let* $\varnothing \neq X$ *be a poset and* $\mathcal{A}_X = \{A_\gamma : \gamma \in \Gamma, A_\gamma : X \to X\}$ *be a commutative family of increasing operators of* X. *Assume that property* (1) *and the following hold*:
(3) *if* C *is a chain of* X *not having supremum, then there exists* $\alpha \in \Gamma$ *such that* $A_\alpha(C)$ *has supremum in* X.
Then there exists an element $m_0 \in X$ *such that* $m_0 = \min\{m \in X: m \in \mathrm{Fix}(\mathcal{A}_X) \cap X^+(u)\}$, *where* $X^+(u) = \{x \in X : x \geq u\}$.

2. Results

Theorem 2 was used in [5] (resp. [6]) for establishing generalizations of results of R.M. Dačič [1] (resp. [2]). Now we establish the following result:

Theorem 3. *Let* $\varnothing \neq X$ *be a poset and* $\mathcal{A}_X = \{A_\gamma : \gamma \in \Gamma, A_\gamma : X \to X\}$ *be a commutative family of increasing operators of* X. *Assume that property* (1) *and the following hold*:
(4) *if* C *is a chain of* X *containing* u *and not having supremum, then there exists* $\alpha \in \Gamma$ *such that* $A_\alpha(C)$ *has supremum in* X.
Then there exists an element $m_0 \in X$ *such that* $m_0 = \min\{m \in X: m \in \mathrm{Fix}(\mathcal{A}_X) \cap X+(u)\}$.

Proof. Let $S = \{C : C$ is a chain of X containing u and $x \leq A\gamma(x)$ for all $\gamma \in \Gamma$ and $x \in C\}$. S is nonempty since it contains the chain $C\gamma(u) = \{u, A_\gamma(u), A_\gamma^2(u), \ldots, A_\gamma^n(u), \ldots\}$ for every $\gamma \in \Gamma$. By Zorn's lemma, there exists in ***S*** a maximal chain C_0. If C_0 has not supremum, property (4) implies $\sup A_\alpha(C_0) = m$ for some $\alpha \in \Gamma$. Since $x \leq A_\gamma(x) \leq m$ for all $\gamma \in \Gamma$ and $x \in C_0$ and bearing in mind that $\mathcal{A}_X$ is a commutative family of increasing operators, we have $x \leq A_\alpha(x) \leq A_\alpha(A_\gamma(x)) = A_\gamma(A_\alpha(x)) \leq A_\gamma(m)$ for all $\gamma \in \Gamma$ and $x \in C_0$. This means that $A_\gamma(m)$ is an upper bound of the chain $A_\alpha(C_0)$ and hence $m \leq A_\gamma(m)$ for all $\gamma \in \Gamma$. Thus $C_0 \cup \{m\}$ is an element of ***S*** and then $m \in C_0$ since C_0 is a maximal chain. This implies that $A_\alpha(m) \leq m$ and therefore $m = A_\alpha(m)$.We also have that $A_\beta(m) \leq A_\beta(A_\gamma(m)) = A_\gamma(A_\beta(m)$ for all $\gamma, \beta \in \Gamma$ and hence $C_0 \cup \{A_\beta(m)\}$ is an element of ***S*** for every $\beta \in \Gamma$. Thus $A_\beta(m) \in C_0$ since C_0 is a maximal chain and then $A_\beta(m) = A_\beta(A_\alpha(m)) = A_\alpha(A_\beta(m)) \leq m$ and therefore $m = A_\beta(m)$, that is $m \in \mathrm{Fix}(\mathcal{A}_X)$. Since no confusion can arise, now suppose that C_0 has supremum m and similarly then it is proved that $m \in \mathrm{Fix}(\mathcal{A}_X)$. Clearly the set $\mathrm{Fix}(\mathcal{A}_X) \cap X^+(u)$ is nonempty since it contains m. Then we can consider the set $W = \{x \in X : u \leq x \leq A_\gamma(x) \leq m$ for all $\gamma \in \Gamma$ and for all $m \in \mathrm{Fix}(\mathcal{A}_X) \cap X^+(u)\}$ which is nonempty since it contains the elements of the chain $C_\gamma(u)$ for every $\gamma\in\Gamma$. Then we define the set ***S'*** = {C : C is a chain of W containing u} which is nonempty since it contains the chain $C_\gamma(u)$ for every $\gamma\in\Gamma$. The Zorn's lemma guarantees the existence in ***S'*** of a maximal chain D_0. As above one can prove that either $\sup D_0$ (if such supremum exists) or $\sup A_\alpha(D_0)$ for some $\alpha\in\Gamma$ (if $\sup D_0$ does not exist) is an element of $Fix(\mathcal{A}_X)$. By calling such element with m_0, we have clearly $m_0 = \min\{m\in X: m\in Fix(\mathcal{A}_X)\cap X^+(u)\}$. ■

Obviously property (4) is more general of (3) and (2) and then Theorem 3 extends Theorems 1 and 2. Thus Theorem 3 is useful in the situations in which Theorem 1 is not applicable as it is shown in the following example:

Example 1. Let $X = [0,1]-\{⅓\}$ with its natural ordering and $\mathcal{A}_X = \{A_1,A_2\}$ where $A_1, A_2 : X\to X$ are defined as $A_1(x) = x$, $A_2(x) = 1/6$ if $x\in[0,⅓)$ and $A_1(x) = (x+2)/3$, $A_2(x) = x$ if $x\in(⅓,1]$. Note that $\mathcal{A}_X$ is a commutative family of increasing operators of X and we point out that an element u verifies property (1) if and only if $u\in[0,1/6]\cup(⅓,1]$. Consider $u\in [0,1/6]$ and the unique chains of X containing u and not having supremum are those chains C such that $u\in C$ and $\sup C = ⅓\notin X$ as supremum in the reals. Then property (2) of Theorem 1 is not satisfied whereas property (4) of Theorem 3 holds since $\sup A_2(C) = 1/6$. Further, $\mathcal{A}_X = \{1/6,1\}$ and we have that $m_0= 1/6$ since $Fix(\mathcal{A}_X)\cap X^+(u) = \{1/6,1\}$ if $u\in[0, 1/6]$ and $m_0 = 1$ since $Fix(\mathcal{A}_X)\cap X^+(u) = \{1\}$ if $u\in(⅓,1]$.

Condition (4) is essential in Theorem 3 otherwise the conclusion could be false, even if we consider a unique operator $A : X\to X$, as it is proved in the following example:

Example 2. Let $X=[0,1]-\{½\}$ with its natural ordering and define $A: X\to X$ as $A(x) = (x+1)/3$ for all $x\in X$. Assume, for instance, $u = 0$ and then $u \le ⅓ = A(u)$. By setting $c_n = (n - 2)/2n$ with n positive integer greater than 2, take the chain $C = \{c_n : n = 2,3,\dots\} \subseteq [0, ½)$ containing $u = 0$ and such that $\sup C = ½\notin X$, thus C has not supremum in X. It is easily seen that $\sup A(C) = \sup\{(3n - 2)/6n: n = 2,3,\dots\} = ½\notin X$. Then all the assumptions of Theorem 3 hold except condition (4) but $\mathcal{A}_X =\varnothing$ because $A(x) = x$ iff $x = ½\notin X$.

As in [4], the following theorem also holds:

Theorem 4. *Let* $\varnothing \neq X$ *be a poset and* $\mathcal{A}_X = \{A_\gamma: \gamma\in\Gamma, A_\gamma: X \to X\}$ *be a commutative family of increasing operators of* X. *Assume that*

(5) *there exists a point* $u\in X$ *such that* $u \ge A_\gamma(u)$ *for every* $\gamma\in\Gamma$,

(6) *if* C *is a chain of* X *containing* u *and not having infimum, then there exists* $\alpha\in\Gamma$ *such that* $A_\alpha(C)$ *has infimum in* X.

Then there exists an element $m_0 \in X$ *such that* $m_0 = \max\{m \in X: m\in Fix(\mathcal{A}_X)\cap X^-(u)\}$, *where* $X^-(u) = \{x\in X : x\le u\}$.

Now we recall some definitions from lattice theory . Let $(X, \le)$ be a poset and S be a nonempty subset of X. S is a complete upper - semilattice (resp. lower –

semilattice) if any nonempty subset of S has supremum (resp. infimum) in S. S is a complete semilattice if it is both a complete upper-semilattice and a complete lower-semilattice. S is a chain complete upper - semilattice (resp. lower - semilattice) if a nonempty chain of S has supremum (resp. infimum) in S. S is a chain complete semilattice if it is both a chain complete upper- semilattice and a chain complete lower – semilattice.

By using Theorem 3 (resp. Theorem 4), we deduce the following result:

Corollary 5. *Let* $\emptyset \neq X$ *be a chain complete upper-semilattice (resp. lower – semilattice) and* $\mathcal{A}_X = \{A_\gamma: \gamma \in \Gamma, A_\gamma: X \to X\}$ *be a commutative family of increasing operators of* X. *If there exists a point* $u \in X$ *such that* $u \leq A_\gamma(u)$ (resp. $u \geq A_\gamma(u)$) *for every* $\gamma \in \Gamma$, *then* $\mathrm{Fix}(\mathcal{A}_X)$ *is a nonempty chain complete upper-semilattice (resp. lower – semilattice) and there exist maximal (resp. minimal) common fixed points for the family* $\mathcal{A}_X$. *Further, if* X *is a chain complete semilattice, then* $\mathrm{Fix}(\mathcal{A}_X)$ *is such as well.*

Proof. We prove the thesis only in the case that X is a chain complete upper-semilattice (the other case is treated similarly). Since any chain of X has supremum, all the assumptions of Theorem 3 are satisfied and hence $\mathrm{Fix}(\mathcal{A}_X) \neq \emptyset$. Let S be a nonempty chain of $\mathrm{Fix}(\mathcal{A}_X)$ and let s be the supremum of S in X. Then we have $x = A_\gamma(x) \leq A_\gamma(s)$ for every $\gamma \in \Gamma$ and $x \in X$, which implies $s \leq A_\gamma(s)$ for every $\gamma \in \Gamma$. Hence there exists an element $s_0 \in \mathrm{Fix}(\mathcal{A}_X)$ such that $s_0 = \min\{y \in X: y \in \mathrm{Fix}(\mathcal{A}_X) \cap X^+(s)\}$ by Theorem 3, that is s_0 is the supremum of S in $\mathrm{Fix}(\mathcal{A}_X)$. This means that $\mathrm{Fix}(\mathcal{A}_X)$ is a chain complete upper-semilattice. Now let C be a maximal chain of $\mathrm{Fix}(\mathcal{A}_X)$ and we put m = sup C. As above, we have $m \leq A_\gamma(m)$ for every $\gamma \in \Gamma$ and by Theorem 3, we can find $m_0 \in X$ such that $m_0 = \min\{y \in X: y \in \mathrm{Fix}(\mathcal{A}_X) \cap X^+(m)\}$. Now $C \cup \{m_0\}$ is a chain of $\mathrm{Fix}(\mathcal{A}_X)$ and hence $m_0 \in C$ since C is maximal, thus $m_0 \leq m$ and then $m_0 = m$. It is plain that m is a maximal element of $\mathrm{Fix}(\mathcal{A}_X)$. ■

Concerning Corollary 5, we exhibit the following example:

Example 3. Let $X = [1, \infty)$ with its natural ordering. Then X is a chain complete lower - semilattice and let $\Gamma = \{1, 2, \ldots, n, \ldots\}$. Define $A_\gamma : X \to X$ as $A_\gamma(x) = (x + \gamma)/(\gamma + 1)$ for all $\gamma \in \Gamma$ if $x \in [1,2]$, $A_\gamma(x) = x$ for all $\gamma \in \Gamma$ if $x \in (2, \infty)$ and let $\mathcal{A}_X$ be the family of such operators. Every operator A_γ is increasing and since $A_\alpha(A_\beta(x)) = A_\beta(A_\alpha(x)) = (x+\alpha+\beta+\alpha\beta)/(\alpha+1)(\beta+1)$ for all $\alpha,\beta \in \Gamma$ if $x \in [1,2]$, $A_\alpha(A_\beta(x)) = A_\beta(A_\alpha(x)) = x$ for all $\alpha,\beta \in \Gamma$ if $x \in (2, \infty)$, $\mathcal{A}_X$ is a commuting family over X. We observe that $\gamma u \geq u$ if $u \in [1,2]$ and $\gamma \in \Gamma$, thus $u \geq A_\gamma(u)$ for all $\gamma \in \Gamma$ and $u \in [1,2]$. Moreover $u = A_\gamma(u)$ for all $\gamma \in \Gamma$ and $u \in (2, \infty)$. Then all the conditions of Corollary 5 are satisfied and we have that $\mathrm{Fix}(\mathcal{A}_X) = \{1\} \cup (2, \infty)$ with 1 as minimum common fixed point.

Analogously we can prove the following theorem:

Corollary 6. Let $\emptyset \neq X$ be a complete upper-semilattice (resp. lower –

semilattice) and $\mathcal{A}_X = \{A_\gamma: \gamma \in \Gamma, A_\gamma: X \to X\}$ be a commutative family of increasing operators of X. If there exists a point $u \in X$ such that $u \le A_\gamma(u)$ (resp. $u \ge A_\gamma(u)$) for every $\gamma \in \Gamma$, then $Fix(\mathcal{A}_X)$ is a nonempty complete upper-semilattice (resp. lower – semilattice) and there exists a maximum (resp. minimum) common fixed point for the family $\mathcal{A}_X$. Further, if X is a complete semilattice then $Fix(\mathcal{A}_X)$ is such as well and has universal bounds.

Remark 1. Corollaries 5 and 6 extend Theorems 6 and 5 of Dacič [2], respectively.

We illustrate Corollary 6 with the following example:

Example 4. Let $X = [0,1]$ be with its ordering natural. Then X is a complete semilattice and let $\Gamma = \{1, 2,\dots, n,\dots\}$. Define $A_\gamma : X \to X$ as $A_\gamma(x) = (x + \gamma)/(2\gamma + 1)$ for all $\gamma \in \Gamma$ if $x \in [0,½]$ and $A_\gamma(x) = x$ if $x \in (½,1]$ and let $\mathcal{A}_X$ be the family of such operators. We point out that every operator A_γ is increasing and moreover we have that $A_\alpha(A_\beta(x)) = A_\beta(A_\alpha(x)) = (x+\alpha+\beta+2\alpha\beta)/(2\alpha+1)(2\beta+1)$ for all $\alpha,\beta \in \Gamma$ and $x \in [0,½]$, $A_\alpha(A_\beta(x)) = A_\beta(A_\alpha(x)) = x$ for all $\alpha,\beta \in \Gamma$ and $x \in (½,1]$, that is $\mathcal{A}_X$ is a commuting family over [0,1]. We note that $2\gamma u \le \gamma$ if $u \in [0,½]$ and $\gamma \in \Gamma$, thus $u \le A_\gamma(u)$ for all $\gamma \in \Gamma$ and $u \in [0,½]$. Moreover $u = A_\gamma(u)$ for all $\gamma \in \Gamma$ and $u \in (½,1]$. Then all the conditions of Corollary 6 are satisfied and we have that $Fix(\mathcal{A}_X) = [½,1]$ is a complete semilattice with minimum ½ and maximum 1.

Remark 2. The commutativity is necessary in all theorems here given otherwise $Fix(\mathcal{A}_X) = \emptyset$ as proved in the following example borrowed from Dacič [2], even if ordered finite spaces are involved. Indeed, let $X = \{a,b,c,d\}$ with (total) ordering defined from $a \le b \le c \le d$ and $\mathcal{A}_X = \{A_1,A_2\}$ where $A_1, A_2 : X \to X$ are defined as $A_1(a) = A_1(b) = c$, $A_1(c) = A_1(d) = d$, $A_2(a) = A_2(b) = b$, $A_2(c) = A_2(d) = c$. Of course X is a (chain) complete semilattice, A_1 and A_2 are increasing operators of X which do not commute because $A_1(A_2(c)) = A_1(c) = d \ne c = A_2(d) = A_2(A_1(c))$ and such that $Fix(\mathcal{A}_X) = \emptyset$.

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