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Weakly ALC Spaces

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Abstract

In this paper we introduce the notion of weakly ALC spaces as spaces in which every almost Lindelöf subset is closed. Weakly ALC spaces are placed between ALC spaces and LC spaces. Several properties, mapping properties of such spaces are studied extensively, it is also shown that in a regular space X if every point has a weakly ALC neighborhood, then X is weakly ALC.

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1. Introduction

A space X is called almost Lindelöf (see [6]) if any open cover of X has a countable subfamily whose union is dense in X, a subset of a space X is called almost Lindelöf if it is almost Lindelöf as a subspace. A subset A of a space (X, τ) is said to be almost Lindelöf in X (relative to X) [7] if any τ-open cover of A has a countable subfamily, the closure of the union of whose members contains A. It was pointed out in [7] that if A is an almost Lindelöf subspace of a space X, then A is almost Lindelöf in X but not conversely.

A space X is called L-closed (see [4]) if every Lindelöf subset of X is closed. We will denote L-closed spaces by LC. X is called ALC (see [7]) if every subset of X which is almost Lindelöf in X is closed.

A subset A of a space X is called regular open if $A = \operatorname{Int}\overline{A}$, and regular closed if $X \setminus A$ is regular open, or equivalently, if $A = \overline{\operatorname{Int} A}$; it is well known that $\operatorname{Int}\overline{A}$ (respectively, $\overline{\operatorname{Int} A}$) is a regular open (respectively, regular closed) subset of X for every subset A of a space X. The family of all regular open subsets of a space (X, τ) forms a base for a coarser topology τ_s on X, (X, τ) is called semi-regular if $\tau = \tau_s$, the space (X, τ_s) will be denoted by X_s and X_s is called the semi-regularization of X. One observes that X_s is Hausdorff if and only if X is Hausdorff.

A space X is called a P-space (see [3]) if the countable intersection of open subsets of X is open and a weak P-space (see [1]) if the countable intersection of regular open subsets of X is regular open, or equivalently, if X_s is a P-space.

A function f from a space X into a space Y is called almost open (see [9]) if $f^{-1}(\overline{U}) \subset \overline{f^{-1}(U)}$ whenever U is open in Y. Clearly, every open function is almost open.

Throughout this paper, no separation axioms are assumed. N denotes the set of natural numbers. For the concepts not defined here we refer the reader to Engelking [2].

In concluding this section, we recall the following facts for their importance in the material of our paper.

Proposition 1.1 ([7]) :

(i) The countable union of subspaces of a space X each of which almost Lindelöf in X is almost Lindelöf in X.

(ii) If $f : X \to Y$ is a continuous function and A is almost Lindelöf in X, then f(A) is almost Lindelöf in Y.

Corollary 1.2 :

(i) The countable union of subspaces of a space X each of which almost Lindelöf is almost Lindelöf.

(ii) If $f : X \to Y$ is a continuous function and A is an almost Lindelöf subset of X, then f(A) is an almost Lindelöf subset of Y.

Proof:

(i) Let $X = \bigcup_{\alpha \in \Lambda} X_\alpha$, where X_α is an almost Lindelöf subset of X for each $\alpha \in \Lambda$. Then X_α is almost Lindelöf in X for each $\alpha \in \Lambda$. By Proposition 1.1 (i), $\bigcup_{\alpha \in \Lambda} X_\alpha$ is almost Lindelöf in X, that is, X is almost Lindelöf .

(ii) Without loss of generality, we may assume that $A = X$ and $f(A) = Y$ (that is because if $f : X \to Y$ is continuous and $A \subset X$, then $f : A \to f(A)$ is continuous). Suppose that X is almost Lindelöf . Then by Proposition 1.1 (ii), f(X) is almost Lindelöf in Y, that is, Y is almost Lindelöf.

Proposition 1.3 ([7]): A regular closed subset of an almost Lindelöf space X is almost Lindelöf.

2. Weakly ALC Spaces

Definition 2.1: A space X is called weakly ALC if every almost Lindelöf subset of X is closed.

Clearly, every ALC space is weakly ALC and every weakly ALC space is LC, questions concerning the converses are posed at the end of this paper.

Lemma 2.2 ([8]): For a space X, the following are equivalent:

(i) X is a weak P-space.

(ii) The countable union of regular closed subsets of X is closed in X_s.

(iii) The countable union of regular closed subsets of X is closed.

It is clear from the above Lemma that every P-space is a weak P-space.

The following theorem was obtained in [7], we will prove it for the convenience of the reader.

Theorem 2.3 ([7]): Let X be a Hausdorff weak P-space. Then every subset of X which is almost Lindelöf in X is closed in X_s and thus X is ALC.

Proof: Let F be a subset of X which is almost Lindelöf in X and let $x \notin F$. Then for each $y \in F$ there exist two disjoint regular open sets U_y, V_y containing x and y respectively (as X is Hausdorff). Since F is almost Lindelöf in X, there exist $y_1, y_2, ... \in F$ such that $F \subset \overline{\bigcup_{i=1}^{\infty} V_{y_i}}$. Let $U = \bigcap_{i=1}^{\infty} U_{y_i}$. Then U is a regular open neighborhood of x which is disjoint from F (as X is a weak P-space). Hence F is closed in X_s.

Corollary 2.4: Let X be a Hausdorff weak P-space. Then every almost Lindelöf subset of X is closed in X_s and thus X is weakly ALC.

Corollary 2.5 ([4]): Let X be a Hausdorff P-space. Then X is LC.

Remark 2.6: The absence of the Hausdorff condition in Corollary 2.4 does not imply that X is weakly ALC as any uncountable set X with the co-countable topology shows (X is a P-space that is not even LC. However, every countable subset of X is closed).

Example 2.7: Let X be an uncountable set. Choose a point $x_0 \in X$ and consider the topology τ on X generated by having all sets {x} open for $x \neq x_0$, if $x_0 \in G$, then G is open if and only if $X \setminus G$ is countable. Then (X, τ) is a non-discrete Hausdorff P-space, thus it is weakly ALC by Corollary 2.4.

Theorem 2.8: If X is a weakly ALC space in which every regular closed subset A of X is almost Lindelöf, then X is a weak P-space. In particular, a weakly ALC almost Lindelöf space is a weak P-space.

Proof: Let $F = \bigcup_{i=1}^{\infty} F_i$ where each F_i is regular closed. Then by assumption, F_i is almost Lindelöf for each $i \in N$ and thus by Proposition 1.2 (i), F is almost Lindelöf, but X is weakly ALC, so F is closed. Hence by Lemma 2.2, X is a weak P-space. The last part follows from Proposition 1.3.

Corollary 2.9 ([7]): Let X be an almost Lindelöf ALC space. Then X is a weak P-space.

Corollary 2.10: For a Hausdorff almost Lindelöf space X, the following are equivalent:

(i) X is ALC.
(ii) X is weakly ALC.
(iii) X is a weak P-space.

Theorem 2.11 ([4]): For a Hausdorff Lindelöf space X, the following are equivalent:

(i) X is LC.
(ii) X is a P-space.

Corollary 2.12: For a Hausdorff Lindelöf space, the following are equivalent:

(i) X is ALC.
(ii) X is weakly ALC.
(iii) X is LC.
(iv) X is a P-space.
(v) X is a weak P-space.

The following result can be easily verified.

Proposition 2.13: For a hereditarily almost Lindelöf space X, the following are equivalent:

(i) X is ALC.
(ii) X is weakly ALC.
(iii) X is a countable discrete space.

Proposition 2.14 ([4]): For a hereditarily Lindelöf space X, the following are equivalent:

(i) X is LC.
(ii) X is a countable discrete space.

Corollary 2.15: For a hereditarily Lindelöf space X, the following are equivalent:

(i) X is ALC.
(ii) X is weakly ALC.
(iii) X is LC.
(iv) X is a countable discrete space.

The following lemma was obtained in [5]. We will, however, prove it for the convenience of the reader.

Lemma 2.16: Let $f : X \to Y$ be an open and closed function from a space X onto a space Y such that $f^{-1}(y)$ is Lindelöf for each $y \in Y$. Then $f^{-1}(F)$ is almost Lindelöf for every almost Lindelöf subset F of Y.

Proof: Without loss of generality, we may assume that $F = Y$ and $f^{-1}(F) = X$ (that is because if $f : X \to Y$ is an open and closed onto function such that $f^{-1}(y)$ is Lindelöf for each $y \in Y$ and $F \subset Y$, then $f : f^{-1}(F) \to F$ is an open and closed onto function such that $f^{-1}(y)$ is Lindelöf for each $y \in F$). Let $\underset{\sim}{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Then for each $y \in Y$, $\underset{\sim}{U}$ has a countable subcollection $\underset{\sim}{U}^{y} = \{U^{y}_{\alpha_j} : j \in N\}$ such that $f^{-1}(y) \subset \bigcup_{j=1}^{\infty} U^{y}_{\alpha_j} = V_y$. Let $W_y = Y \setminus f(X \setminus V_y)$. Then W_y is an open neighborhood of y. Since Y almost Lindelöf , there exist $y_1, y_2, \ldots \in Y$ such that $Y = \overline{\bigcup_{i=1}^{\infty} W_{y_i}}$. Since f is open, it is almost open and thus $X = f^{-1}\left(\overline{\bigcup_{i=1}^{\infty} W_{y_i}}\right) \subset \overline{f^{-1}\left(\bigcup_{i=1}^{\infty} W_{y_i}\right)} = \overline{\bigcup_{i=1}^{\infty} f^{-1}(W_{y_i})} \subset \overline{\bigcup_{i=1}^{\infty} V_{y_i}} = \overline{\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty} U^{y_i}_{\alpha_j}}$. Hence $X = \overline{\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty} U^{y_i}_{\alpha_j}}$ and therefore X is almost Lindelöf .

Theorem 2.17: Let f be an open and closed function from a space X onto a space Y such that $f^{-1}(y)$ is Lindelöf for each $y \in Y$. Then Y is weakly ALC whenever X is weakly ALC.

Proof: Follows from Lemma 2.16.

Theorem 2.18: Let f be a continuous one to one function from a space X into a weakly ALC space Y. Then X is weakly ALC.

Proof: Follows from Corollary 1.2 (ii).

Corollary 2.19: Being weakly ALC is hereditary.

Proof: Let A be a non-empty subset of a weakly ALC space X. Then the inclusion function $i : A \to X$ is continuous and one to one. Thus by Theorem 2.18, A is weakly ALC.

Corollary 2.20: $\oplus X_\alpha$ is weakly ALC if and only if X_α is weakly ALC for each

.

Proof: "$\Rightarrow$" Follows from Corollary 2.19.

"$\Leftarrow$" Suppose that X_α is a weakly ALC space for each . We want to show that $\oplus X_\alpha$ is weakly ALC. Let F be an almost Lindelöf subset of X. Then $F \cap X_\alpha$ is a regular closed subset of F and thus almost Lindelöf by Proposition 1.3, but X_α is weakly ALC, so $F \cap X_\alpha$ is closed in X_α for each $\alpha \in \Lambda$, that is, F is closed in $\oplus X_\alpha$. Hence $\oplus X_\alpha$ is weakly ALC.

Theorem 2.21: Let X and Y be two spaces such that X is semi-regular in which every regular closed subset is almost Lindelöf, Y is weakly ALC. Then any continuous one to one function from X to Y is closed. In particular, any continuous one to one function from a semi-regular almost Lindelöf space X to a weakly ALC space Y is closed.

Proof: Let F be a closed subset of X. Since X is semi-regular, it follows that F is the intersection of regular closed sets $F_\alpha, \alpha \in \Lambda$. By assumption, F_α is almost Lindelöf for each $\alpha \in \Lambda$. Since f is continuous, it follows by Corollary 1.2 (ii) that $f(F_\alpha)$ is almost Lindelöf for each $\alpha \in \Lambda$. Since Y is weakly ALC, it follows that $f(F_\alpha)$ is closed in Y for each $\alpha \in \Lambda$. Finally since f is one to one, it follows that $f(F)$ is the intersection of closed sets $f(F_\alpha)$ in Y and therefore $f(F)$ is closed in Y. The last part follows from Proposition 1.3.

Corollary 2.22 ([7]): A continuous one to one function from a semi-regular almost Lindelöf space X to an ALC space Y is closed.

Theorem 2.23: Let X be a weakly ALC space and Y be any space. If $f : X \to Y$ is a function whose graph G_f is a semi-regular subspace in which every regular closed subset is almost Lindelöf, then f is continuous. In particular, any function having a weakly ALC domain and a semi-regular almost Lindelöf graph is continuous.

Proof: Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the projection functions and let $p_X^* = p_X\big|_{f}$. Then it follows from Theorem 2.21 that p_X^* is a closed function. Thus $(p_X^*)^{-1}$ is continuous and therefore $f = p_Y o (p_X^*)^{-1}$ is continuous too. The last part follows from Proposition 1.3.

Corollary 2.24 ([7]): Any function having an ALC domain and a semiregular almost Lindelöf graph is continuous.

Theorem 2.25: If in a regular space X, every point has a weakly ALC neighborhood, then X is weakly ALC.

Proof: Let F be an almost Lindelöf subset of X, $x \notin F$. Since the property of being weakly ALC is hereditary (Corollary 2.19), it follows by assumption that there exists an open weakly ALC neighborhood of x, but X is regular, so there exists an open set V such that $x \in V \subset \overline{V} \subset U$. Now $\overline{V \cap F}^F$ is a regular closed subset of F, but F is almost Lindelöf, so by Proposition 1.3, $\overline{V \cap F}^F$ is almost Lindelöf, but U is weakly ALC, so $\overline{V \cap F}^F$ is closed in U and thus $V \cap (U \setminus \overline{V \cap F}^F)$ is an open neighborhood of x which is disjoint from F, thus $x \notin \overline{F}$. Hence F is closed and therefore X is weakly ALC.

Finally we pose the following questions:

Question 1: Does there exist a regular LC space which is not weakly ALC ?

Question 2: Does there exist a regular weakly ALC space which is not ALC ?

Question 3: What is a suitable condition “P” such that a space X is weakly ALC if and only if X is LC and satisfies “P”?

Question 4: What is a suitable condition “P” such that a space X is ALC if and only if X is weakly ALC and satisfies “P”?

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