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On the Product of Two Lindelöf Spaces

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Abstract

In [9], Telgarsky showed that the product of a Lindelöf C-scattered space with a Lindelöf space is Lindelöf. In this paper we generalize this result as follows: If X and Y are Lindelöf spaces, X is W_Y-scattered, then $X \times Y$ is Lindelöf.

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1. Preliminaries

In this section some essential definitions and basic facts are introduced. Throughout this paper all spaces are assumed to be regular. If A and B are subsets of a space X such that $A \subset B$, then $\overline{A}^{B}$ denotes the closure of A in B. N denotes the set of natural numbers, the phrase clopen stands for open and closed. For the concepts not defined here we refer the reader to Engelking [1].

Definition 1.1 ([2]):

(i) A subset of a space X is said to be ω-closed if it contains all of its condensation points. The complement of an ω-closed set is said to be ω-open.

(ii) A function f from a space X onto a space Y is said to be ω-closed if it maps closed sets onto ω-closed sets.

(iii) A space X is said to be a P*-space if the intersection of countably many open sets in an ω-open set.

Theorem 1.2 ([4]): Let X be a Lindelöf space. Then X with the topology consisting of ω-open subsets is Lindelöf.

Theorem 1.3 ([2]): If f is an ω-closed function from a space X onto a space Y such that $f^{-1}(y)$ is Lindelöf for each $y \in Y$, then X is Lindelöf whenever Y is so.

Definition 1.4 ([3]). Let X and Y be two spaces. A subset A of X is said to be strongly (weakly) placed in $X \times Y$ if for each $y \in Y$, whenever $A \times \{y\} \subset G$, where G is open in $X \times Y$, there exists an open set $U \subset X$ and open (ω-open) set $V \subset Y$ such that $A \times \{y\} \subset U \times V \subset G$.

Theorem 1.5 ([3]): Let X and Y be two spaces. Then X is weakly placed in X x Y if and only if the projection function $\pi : X \times Y \to Y$ is ω-closed.

Definition 1.6 ([7]): A space X is said to be perfectly zero-dimensional if every open cover of X has a pairwise disjoint open refinement.

Theorem 1.7 ([6]): A space X is perfectly zero-dimensional if and only if it is paracompact and strongly zero-dimensional.

Theorem 1.8 ([5]): A space X is paracompact if and only if there is a perfectly zero-dimensional space $\dot{X}$ and a perfect function from $\dot{X}$ onto X.

Definition 1.9 ([8]): A space X is said to be C-scattered if every non-empty closed subspace A of X has a point $a \in A$ with a compact neighbor-hood in A.

2. A Product Theorem

Definition 2.1: Let X be a paracompact space and Y be any space. Then X is said to be S_Y-scattered (W_Y-scattered) if every non-empty closed subspace A of $\dot{X}$ has a point $a \in A$ with an open neighborhood U in A such that $\overline{U}$ is strongly (weakly) placed in $\dot{X} \times Y$.

Obviously, if X is S_Y-scattered, then it is W_Y-scattered. On the other hand, if X is C-scattered, then X is S_Y-scattered for every space Y. However, if X is S_Y-scattered (W_Y-scattered), then X need not be C-scattered as the following example shows.

Example 2.2: Let S be the Sorgenfrey line and Y be any P-space (P^*-space).Then S is S_Y-scattered (W_Y-scattered). However S is not C-scattered.

Theorem 2.3: Let X, Y be two spaces and let A be a subset of X. Then:

(i) If A is strongly (weakly) placed in $X \times Y$, then A is strongly (weakly) placed in $B \times Y$ for every subset B of X which contains A.

(ii) If A is strongly (weakly) placed in $X \times Y$ and B is a clopen subset of X, then $A \cap B$ is strongly (weakly) placed in $X \times Y$.

(iii) If A is strongly (weakly) placed in $B \times Y$ where B is open in X, then A is strongly (weakly) placed in $X \times Y$.

Proof: Straightforward.

Theorem 2.4: Let X and Y be Lindelöf spaces such that each point $x \in X$ has an open neighborhood U_x whose closure is weakly placed in $X \times Y$. Then $X \times Y$ is Lindelöf.

Proof: Follows directly from Theorems 1.3, 1.5, and 2.3 (i).

The following three theorems can be easily obtained and therefore given without proofs.

Theorem 2.5: Let X and Y be two spaces where X has the property that each $x \in X$ has an open neighborhood U_x such that $\overline{U}_x \times Y$ is Lindelöf. Then any F_σ subset A of X has also the above property.

Theorem 2.6: Let X, Y, and Z be three spaces and let $f : X \to Y$ be a perfect onto function. If each $x \in X$ has an open neighborhood U_x such that $\overline{U}_x \times Z$ is Lindelöf, then each $y \in Y$ has an open neighborhood U_y such that $\overline{U}_y \times Z$ is Lindelöf and visa versa.

Theorem 2.7: Let $f : X \to Y$ be a perfect function and $g : Z \to Z$ be the identity function. If $A \subset Y$ is weakly placed in $Y \times Z$, then $f^{-1}(A)$ is weakly placed in $X \times Z$.

Theorem 2.8: Let X, Y be Lindelöf spaces and A be a Lindelöf subset of X which is weakly placed in $X \times Y$. If every point $x \in X - A$ has an open neighborhood U_x in $X \tilde{} A$ such that $\overline{U}_x^{X-A} \times Y$ is Lindelöf, then $X \times Y$ is Lindelöf.

Proof: Follows from Theorems 1.2 and 2.5.

The following three theorems are immediate consequences of Theorems 1.7 and 1.8.

Theorem 2.9: A space X is a Lindelöf zero-dimensional space if and only if every open cover has a countable pairwise disjoint open refinement.

Theorem 2.10: A space X is Lindelöf if and only if there exists a Lindelöf zero-dimensional space $\dot{X}$ and a perfect function f from $\dot{X}$ onto X.

Theorem 2.11: Let X be a Lindelöf space and Y by any space. Then $X \times Y$ is Lindelöf if and only if $\dot{X} \times Y$ is Lindelöf.

Theorem 2.12: Let X and Y be Lindelöf spaces and let A be a closed subset of X such that each $x \in X$ has an open neighborhood U_x in A whose closure is weakly placed in $X \times Y$ and for each $y \in X - A$ there is an open neighborhood U_y in $X \tilde{} A$ such that $\overline{U_y}^{X-A} \times Y$ is Lindelöf. Then X x Y is Lindelöf.

Proof: By Theorems 2.3 (ii), 2.6, 2.7, 2.10, and 2.11 we may assume that X is a Lindelöf zero-dimensional space. By Theorem 2.9, $\{U_x : x \in A\}$ has a countable pairwise disjoint open refinement $\underset{\sim}{A} = \{A_n : n \in N\}$. Let $A_n = V_n \cap A$ where V_n is open in X and let $\underset{\sim}{V} = \{V_n : n \in N\} \cup \{X - A\}$. Then $\underset{\sim}{V}$ has a countable pairwise disjoint open refinement $\underset{\sim}{M} = \{M_i : i \in N\}$. Thus for each i, M_i intersects at most one member of $\underset{\sim}{A}$, say A_{n_i}. Now it follows from Theorem 2.7 that $M_i \cap A_{n_i}$ is weakly placed
in $M_i \times Y$, on the other hand, $M_i - A_{n_i}$ is a closed subset of $X \tilde{} A$, thus by assumption on $X \tilde{} A$ it follows from Theorems 2.5 and 2.8 that $M_i \times Y$ is Lindelöf. Hence $X \times Y = \bigcup_{i=1}^{\infty} (M_i \times Y)$ is Lindelöf.

Theorem 2.13: Let X and Y be two Lindelöf spaces. If X is W_Y-scattered, then $X \times Y$ is Lindelöf.

Proof: By Theorems 2.10 and 2.11 we may assume that X is a Lindelöf zero-dimensional space. Denote by $X^{(1)} = \{y \in X: y$ has no open neighborhood whose closure is weakly placed in $X \times Y\}$. If $\beta = \alpha + 1$, define $X^{(\beta)} = (X^{(\alpha)})^{(1)}$. If β is a limit ordinal, define $X^{(\beta)} = \bigcap_{\alpha<\beta} X^{(\alpha)}$. Since X is W_Y-scattered, it is easy to see that $X^{(\beta)} = \phi$ for some ordinal □.

We want to prove that if $X^{(\beta)} = \phi$, then $X \times Y$ is Lindelöf.

If $X^{(1)} = \phi$, then $X \times Y$ is Lindelöf by Theorem 2.4. Suppose that the result holds for all $\alpha < \beta$. To complete the induction we have the following two cases:

Case 1: If $\beta = \alpha + 1$ for some □, then $X^{(\beta)} = \phi$ implies that every point x of $X^{(\alpha)}$ has an open neighborhood M_x in $X^{(\alpha)}$ such that $\overline{M}_x$ is weakly placed in $X \times Y$. Now since X is zero-dimensional, for each $y \in X - X^{(\alpha)}$ there is a clopen set M_y such that $y \in M_y \subset X - X^{(\alpha)}$, again since X is zero-dimensional, it follows by Theorem 2.3 that $M_y^{(\alpha)} = \phi$. Thus by the inductive assumption we get that $M_y \times Y$ is Lindelöf and therefore it follows by Theorem 2.12 that $X \times Y$ is Lindelöf.

Case 2: If $X^{(\beta)} = \bigcap_{\alpha<\beta} X^{(\alpha)} = \phi$, then it follows by Theorem 2.9 that the open cover $\{X - X^{(\alpha)} : \alpha < \beta\}$ has a countable pairwise disjoint open refinement $\{H_n : n \in N\}$. Thus $H_n^{(\alpha)} = \phi$ which implies by the inductive step that $H_n \times Y$ is Lindelöf. Hence $X \times Y = \bigcup_{n=1}^{\infty} (H_n \times Y)$ is Lindelöf.

Corollary 2.14: Let X and Y be two Lindelöf spaces. If X is S_Y-scattered, then $X \times Y$ is Lindelöf.

Corollary 2.15 ([9]): Let X and Y be two Lindelöf spaces. If X is C-scattered, then $X \times Y$ is Lindelöf.

Finally we raise the following two questions:

Question 1: Is it possible to find two spaces X and Y such that X is W_Y-scattered which is not S_Y-scattered ?

Question 2: Is it possible to extend the result of Theorem 2.13 to Lindelöf spaces X and Y such that X satisfies a more general condition?

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