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Homomorphism Theorems in GT-Algebras

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Abstract

We introduce the notion of normal GT-filters in GT-algebras, and we establish construct the quotient GT-algebras via normal GT-filter, and we have the fundamental theorem of homomorphisms for GT-algebras as a consequence.

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1 Introduction

The variety of Tarski algebras was introduced by J. C. Abbott in [2]. These algebras are an algebraic counterpart of the $\{\vee, \rightarrow\}$-fragment of the propositional classical calculus. S. A. Celani ([5]) introduced Tarski algebras with

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a modal operator as a generalization of the concept of Boolean algebra with a modal operator which he researched into these fragments of the algebraic viewpoint. Properties of filters in Tarski algebras were treated by S. A. Celani ([5]) and the authors ([6]). Recently, J. Kim, Y. Kim and E. H. Roh ([6]) considered decompositions and expansions of filters in Tarski algebras, and also they have shown that there is no non-trivial quadratic Tarski algebras on a field X with $|X| \geq 3$. However, we feel that the concept of Tarski algebra is relatively too strong for filters. Kim et al. ([7]) established a new algebra, called a GT-algebra, which is a generalization of Tarski algebra, and gave a method to construct a GT-algebra from a quasi-ordered set. In this paper, we introduce the notion of normal GT-filters in GT-algebras, and we establish construct the quotient GT-algebras via normal GT-filter, and we have the fundamental theorem of homomorphisms for GT-algebras as a consequence.

2 Preliminary Notes

Let us review some definitions and results.

Definition 2.1. [7] *By a generalized Tarski algebra (GT-algebra, for short) we mean an algebra* $(X; \rightarrow, 1)$ *of type* $(2, 0)$ *satisfying the following conditions:*

(T1) $(\forall a \in X)(1 \rightarrow a = a)$.

(T2) $(\forall a \in X)(a \rightarrow a = 1)$.

(T3) $(\forall a, b, c \in X)(a \rightarrow (b \rightarrow c) = (a \rightarrow b) \rightarrow (a \rightarrow c))$.

Given a GT-algebra X, if it satisfies the condition

(T4) $(\forall a, b \in X)((a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a)$,

we call the algebra a *Tarski algebra*. In a Tarski algebra X we can define an order relation $\leq$ by setting $a \leq b$ if and only if $a \rightarrow b = 1$ for any $a, b \in X$. Note that $(X; \leq)$ is a poset ([3]).

A reflexive and transitive relation $\mathcal{R}$ on a set X is called a *quasi-ordering* of X, and the couple $(X, \mathcal{R})$ is called a *quasi-ordered set* ([4]). Note that if X is a GT-algebra, then the relation $\leq$ by setting $x \leq y$ if and only if $x \rightarrow y = 1$ for any $a, b \in X$ is a quasi-ordering of X; with respect to this quasi-ordering 1 is the greatest element of X ([8]).

Example 2.2. [8] *Let* $X := \{a, b, c, 1\}$ *be a set with the following Cayley table:*

$\rightarrow$	a	b	c	1
a	1	1	c	1
b	1	1	c	1
c	1	1	1	1
1	a	b	c	1

Then $(X; \to, 1)$ *is a GT-algebra and*

$$\mathcal{R} := \{(a,a),(a,b),(a,1),(b,a),(b,b),(b,1),(c,a),(c,b),(c,c),(c,1),(1,1)\}$$

is a quasi-ordering of X*, which is not an anti-symmetric relation of* X*.*

Lemma 2.3. [7] *Let* X *be a GT-algebra. Then*

(p1) $(\forall a \in X)(a \leq 1)$.

(p2) $(\forall a, b \in X)(a \leq b \to a)$.

(p3) $(\forall a, b \in X)(a \to (a \to b) = a \to b)$.

(p4) $(\forall a, b \in X)(a \leq (a \to b) \to b)$.

(p5) $(\forall a, b, c \in X)(a \leq b \Rightarrow c \to a \leq c \to b)$.

Definition 2.4. [7] *Let* X *be a GT-algebra. A nonempty subset* F *of* X *is called a GT-filter of* X *if it satisfies the following conditions:*

(F1) $(\forall a, b \in X)(b \in F \Rightarrow a \to b \in F)$.

(F2) $(\forall a, b \in X)(a \to b \in F, a \in F \Rightarrow b \in F)$.

Theorem 2.5. [7] *Let* F *be a nonempty subset of a GT-algebra* X*. Then* F *is a GT-filter of* X *if and only if it satisfies* $1 \in F$ *and* (F2)*.*

Let $\mathcal{R}$ be a relation on a GT-algebra X. Then $\mathcal{R}$ is said to be *compatible* if $(a \to e, b \to f) \in \mathcal{R}$ whenever $(a, b) \in \mathcal{R}$ and $(e, f) \in \mathcal{R}$ for all $a, b, e, f \in X$. A compatible equivalence relation on X is said to be a *congruence* on X.

Let X be a GT-algebra and $K(\neq \emptyset) \subseteq X$. Denote by Θ_K the relation on X given by

$$(a, b) \in \Theta_K \ \text{ iff } \ \mathrm{a \to b \in K} \ \text{ and } \ \mathrm{b \to a \in K}.$$

Theorem 2.6. [7] *Let* K *be a GT-filter of a GT-algebra* X*. Then the relation* Θ_K *is an equivalence relation on* X *and* $[1]_{\Theta_K} = K$*.*

In Theorem 2.6, Θ_K may not be compatible in general, as the following example.

Example 2.7. [7] *Let $X := \{a, b, c, d, 1\}$ be a set with the following Cayley table:*

$\rightarrow$	a	b	c	d	e	f	g	1
a	1	1	1	1	1	1	1	1
b	c	1	c	g	1	1	g	1
c	f	f	1	f	1	f	1	1
d	c	e	c	1	e	1	1	1
e	a	f	c	d	1	f	g	1
f	c	e	c	g	e	1	g	1
g	a	b	c	f	e	f	1	1
1	a	b	c	d	e	f	g	1

Then $(X; \rightarrow, 1)$ is a GT-algebra, and the subset $K := \{d, 1\}$ is a GT-filter of X. Moreover, we can find

$$\Theta_K = \{(a, a), (b, b), (c, c), (d, d), (e, e), (e, 1), (f, f), (g, g), (1, e), (1, 1)\}.$$

It is routine to check that Θ_K is an equivalence relation on X, which is not compatible since $(e, 1) \in \Theta_K$ and $(b, b) \in \Theta_K$, but $(e \rightarrow b, 1 \rightarrow b) = (f, b) \notin \Theta_K$.

3 Main Results

Definition 3.1. *A GT-filter F of a GT-algebra X is said to be normal if it satisfies:*

(F3) $(\forall a, b, c \in X)(a \rightarrow b \in F \Rightarrow (b \rightarrow c) \rightarrow (a \rightarrow c) \in F)$.

Obviously, X and $\{1\}$ are normal GT-filters of X.

Example 3.2. *Let $X := \{a, b, c, 1\}$ be a set with the following Cayley table:*

$\rightarrow$	a	b	c	1
a	1	b	1	1
b	a	1	1	1
c	a	b	1	1
1	a	b	c	1

It is easy to check that $(X; \rightarrow, 1)$ is a GT-algebra, and $\{c, 1\}, \{a, c, 1\}, \{b, c, 1\}$ are normal GT-filters of X. But $F := \{a, 1\}$ is not a normal GT-filter of X since $1 \rightarrow a \in F$ and $(a \rightarrow c) \rightarrow (1 \rightarrow c) = c \notin F$.

Now we construct the quotient GT-algebras via normal GT-filters. Let K be a normal GT-filter of a GT-algebra $(X;\rightarrow,1)$. Then we obtain from Theorem 2.6 that Θ_K is an equivalence relation on X and $[1]_{\Theta_K} = K$.

Let $a,b,c,d \in X$ such that $(a,b) \in \Theta_K$ and $(c,d) \in \Theta_K$. Then we have $a \rightarrow b \in K$ implies $(a \rightarrow c) \rightarrow (b \rightarrow c) \in K$, and $b \rightarrow a \in K$ implies $(b \rightarrow c) \rightarrow (a \rightarrow c) \in K$. Thus we get

$$(a \rightarrow c, b \rightarrow c) \in \Theta_K.$$

Since $c \rightarrow d \in K$ implies $(b \rightarrow c) \rightarrow (b \rightarrow d) \in K$, and $d \rightarrow c \in K$ implies $(b \rightarrow d) \rightarrow (b \rightarrow c) \in K$. Hence we have

$$(b \rightarrow c, b \rightarrow d) \in \Theta_K.$$

We conclude that $(a \rightarrow c, b \rightarrow d) \in \Theta_K$. Therefore Θ_K is a congruence relation on X.

Denote the equivalence class containing a by $[a]_{\Theta_K}$, i.e.,

$$[a]_{\Theta_K} := \{x \in X | (a,x) \in \Theta_K\}.$$

We note that $(a,b) \in \Theta_K$ if and only if $[a]_{\Theta_K} = [b]_{\Theta_K}$.

Denote $X/\Theta_K := \{[a]_{\Theta_K} | a \in X\}$ and define

$$[a]_{\Theta_K} \rightarrow' [b]_{\Theta_K} := [a \rightarrow b]_{\Theta_K}.$$

The operation "$\rightarrow'$" is well-defined, since Θ_K is a congruence relation on X. We claim that $(X/\Theta_K, \rightarrow', [1]_{\Theta_K})$ is a GT-algebra. Clearly

$$[1]_{\Theta_K} \rightarrow' [a]_{\Theta_K} = [a]_{\Theta_K} \text{ and } [\mathrm{a}]_{\Theta_\mathrm{K}} \rightarrow' [\mathrm{a}]_{\Theta_\mathrm{K}} = [1]_{\Theta_\mathrm{K}}$$

for all $[a]_{\Theta_K} \in X/\Theta_K$. Let $[a]_{\Theta_K}, [b]_{\Theta_K}, [c]_{\Theta_K} \in X/\Theta_K$. Then we have

$$\begin{aligned} [a]_{\Theta_K} \rightarrow' ([b]_{\Theta_K} \rightarrow' [c]_{\Theta_K}) &= [a \rightarrow (b \rightarrow c)]_{\Theta_K} \\ &= [(a \rightarrow b) \rightarrow (a \rightarrow c)]_{\Theta_K} \\ &= ([a]_{\Theta_K} \rightarrow' [b]_{\Theta_K}) \rightarrow' ([a]_{\Theta_K} \rightarrow' [c]_{\Theta_K}). \end{aligned}$$

We summarize:

Theorem 3.3. *Let K be a normal GT-filter of a GT-algebra $(X;\rightarrow,1)$. If we define*

$$[a]_{\Theta_K} \rightarrow' [b]_{\Theta_K} := [a \rightarrow b]_{\Theta_K}$$

for all $a,b \in X$, then $(X/\Theta_K;\rightarrow',[1]_{\Theta_K})$ is a GT-algebra, which is called the *quotient GT-algebra* via K.

Now, we state a fundamental theorem of a homomorphism.

Definition 3.4. *Let X, Y be GT-algebras. A mapping $f : X \to Y$ is called an homomorphism if*

$$((\forall a, b \in X)(f(a \to b) = f(a) \to f(b)).$$

A homomorphism f is called a monomorphism (resp., epimorphism) if it is injective (resp., surjective). A bijective homomorphism is called an isomorphism. Two GT-algebras X and Y are said to be isomorphic, written $X \cong Y$, if there exists an isomorphism $f : X \to Y$. For any homomorphism $f : X \to Y$, the set $\{a \in X | f(a) = 0\}$ is called the kernel of f, denoted by $Ker(f)$ and the set $\{f(a) | a \in X\}$ is called the image of f, denoted by $Im(f)$.

Example 3.5. *Let $X := \{a, b, 1\}$ and $Y := \{x, 1\}$ be GT-algebras, whose Cayley tables are as follows.*

$\to$	a	b	1
a	1	1	1
b	a	1	1
1	a	b	1

$\to$	x	1
x	1	1
1	x	1

Define a mapping $f : X \to Y$ by $a \mapsto x, b \mapsto 1, 1 \mapsto 1$. It can be readily check that f is a homomorphism from a GT-algebra X to a GT-algebra Y.

Example 3.5 shows that the cardinal number of a homomorphic image of a finite GT-algebra may not be a factor of the cardinal number of the domain. Obviously, $|X| = 3$ and $|Y| = 2$, but 2 is not factor of 3.

Lemma 3.6. *Let $f : X \to Y$ be a homomorphism from a GT-algebra X to a GT-algebra Y. Then*

(i) $f(1) = 1$.

(ii) $(\forall a, b \in X)(a \leq b \Rightarrow f(a) \leq f(b))$.

Proof. Straightforward. □

The next proposition satisfies an ordinary algebraic homomorphism, whose verification is routine and omitted.

Lemma 3.7. *Let $f : X \to Y$ be a homomorphism from a GT-algebra X to a GT-algebra Y. Then*

(i) *f is epimorphic if and only if $Im(f) = Y$*

(ii) *f is monomorphic if and only if $Ker(f) = \{1\}$*

(iii) *f is isomorphic if and only if the inverse mapping $f^{-1} : Y \to X$ is isomorphic.*

Theorem 3.8. *Let $f : X \to Y$ be a homomorphism from a GT-algebra X onto a Tarski algebra Y. Then $Ker(f)$ is a normal GT-filter of X.*

Proof. Obviously, $1 \in Ker(f)$. Let $a \to b \in ker(f)$ and $a \in ker(f)$. Then $1 = f(a \to b) = f(a) \to f(b) = 1 \to f(b) = f(b)$. Hence $b \in ker(f)$. Let $a \to b \in ker(f)$ Then for any $c \in X$, we have

$$f((b \to c) \to (a \to c)) = 1.$$

Hence we obtain $(b \to c) \to (a \to c) \in Ker(f)$. Therefore $ker(f)$ is a normal GT-filter of X. □

In Example 3.5, X is a GT-algebra, which is not a Tarski algebra since $(a \to b) \to b \neq (b \to a) \to a$, and Y is a Tarski algebra, and f is an epimorphism. Obviously, $Ker(f) = \{b, 1\}$ is a normal GT-filter of X.

Theorem 3.9. (Homomorphism Theorem) *If $f : X \to Y$ is a homomorphism from a GT-algebra X onto a Tarski algebra Y, then the quotient GT-algebra $X/\Theta_{Ker(f)}$ and Y are isomorphic, i.e., $X/\Theta_{Ker(f)} \cong Y$.*

Proof. Define a mapping

$$\mu : X/\Theta_{Ker(f)} \to Y \text{ by } \mu([a]_{\Theta_{Ker(f)}}) = f(a).$$

If $[a]_{\Theta_{Ker(f)}} = [b]_{\Theta_{Ker(f)}}$, then $a \to b, b \to a \in Ker(f)$, and so we get

$$f(a) \to f(b) = 1 = f(b) \to f(a)$$

in Y. Thus we have $f(a) = f(b)$, i.e., $\mu([a]_{\Theta_{Ker(f)}}) = \mu([b]_{\Theta_{Ker(f)}})$. This means that μ is well-defined. Let $[a]_{\Theta_{Ker(f)}}, [b]_{\Theta_{Ker(f)}} \in X/\Theta_{Ker(f)}$ with $[a]_{\Theta_{Ker(f)}} \neq [b]_{\Theta_{Ker(f)}}$. Then $(a, b) \notin \Theta_{Ker(f)}$, and hence

$$\text{either } a \to b \notin Ker(f) \text{ or } b \to a \notin Ker(f).$$

Without loss of generality, we may assume $a \to b \notin Ker(f)$. It follows that $f(a) \to f(b) = f(a \to b) \neq 1$, and hence $f(a) \neq f(b)$. This means that μ is one-one. For any $b \in Y$, there is an $a \in X$ such that $b = f(a)$, since f is onto. Hence $\mu([a]_{\Theta_{Ker(f)}}) = f(a) = b$, which means that μ is onto. Since

$$\begin{aligned}
\mu([a]_{\Theta_{Ker(f)}} \to [b]_{\Theta_{Ker(f)}}) &= \mu([a \to b]_{\Theta_{Ker(f)}}) \\
&= f(a \to b) \\
&= f(a) \to f(b) \\
&= \mu([a]_{\Theta_{Ker(f)}}) \to \mu([b]_{\Theta_{Ker(f)}}),
\end{aligned}$$

μ is a homomorphism. Thus we obtain $X/\Theta_{Ker(f)} \cong Y$, completing the proof. □

Theorem 3.10. *Let X and Y be GT-algebras and Z be a Tarski algebra, and let $h : X \to Y$ be an epimorphism and $g : X \to Z$ be a homomorphism. If $Ker(h) \subseteq Ker(g)$, then there is a unique homomorphism $f : Y \to Z$ satisfying $f \circ h = g$.*

Proof. For any $b \in Y$, there exists an $a \in X$ such that $b = h(a)$. Given an element a, we put $c := g(a)$. Define a mapping

$$f : Y \to Z \text{ by } f(b) = c.$$

To prove that f is well-defined and $f \circ h = g$. If $b = h(a_1) = h(a_2), a_1, a_2 \in X$, then $0 = h(a_1) \to h(a_2) = h(a_1 \to a_2)$. Hence $a_1 \to a_2 \in Ker(h)$. Since $Ker(h) \subseteq Ker(g)$, we have $0 = g(a_1 \to a_2) = g(a_1) \to g(a_2)$. Similarly, we get $g(a_2) \to g(a_1) = 0$. Thus $g(a_2) = g(a_1)$. This means that f is well-defined. Clearly $g(a) = f(h(a))$ for any $a \in X$. To show that f is a homomorphism. Let $b_1, b_2 \in Y$. For any $a_1, a_2 \in X$ such that $b_1 = h(a_1), b_2 = h(a_2)$, we have $f(b_1 \to b_2) = f(h(a_1) \to h(a_2)) = f(h(a_1 \to a_2)) = g(a_1 \to a_2) = g(a_1) \to g(a_2) = f(h(a_1)) \to f(h(a_2)) = f(b_1) \to f(b_2)$. Hence f is a homomorphism. The uniqueness of f follows directly from the fact that h is an epimorphism. □

Theorem 3.11. *Let X, Y and Z be GT-algebras, and let $g : X \to Z$ be a homomorphism and $h : Y \to Z$ be a monomorphism with $Im(g) \subseteq Im(h)$, then there is a unique homomorphism $f : X \to Y$ satisfying $h \circ f = g$.*

Proof. For each $a \in X$, $g(a) \subseteq Im(g) \subseteq Im(h)$. Since h is a monomorphism, there exists a unique $b \in Y$ such that $h(b) = g(a)$. Define a map

$$f : X \to Y \text{ by } f(a) = b.$$

Then $h \circ f = g$. We show that f is a homomorphism. If $a_1, a_2 \in X$, then $g(a_1 \to a_2) = h(f(a_1 \to a_2))$. On the other hand, since g is a homomorphism, $g(a_1 \to a_2) = g(a_1) \to g(a_2) = h(f(a_1)) \to h(f(a_2)) = h(f(a_1) \to f(a_2))$. Hence $h(f(a_1 \to a_2)) = h(f(a_1) \to f(a_2))$. Since h is a monomorphism, we have $f(a_1 \to a_2) = f(a_1) \to f(a_2)$. The uniqueness of f follows from the fact that h is a monomorphism. □

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