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Existence of Solution of an Optimal Inventory Equation with Unbounded Time Period

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Abstract

In this paper existence of the solution of the optimal inventory equation resulting from a multistage allocation process with unbounded time period is studied and the existence of the solution is proved.

Keywords: Multi stage allocation process, Inventory control programming, Fixed point

1 Introduction

The successive approximation is a powerful analytic tool to prove the existence of the optimal policy of functional equations arising in optimization problems. R.Bellman[4],[5] explored this area of optimization by investigating the existence and behavior of the solution of various types of functional equations in game theory, inventory control problems, dynamic programming, bottleneck problems etc. Bhakta and Mitra [3] proved some existence theorems for general multistage allocation process using dynamic programming method. Panda and Senapati [2] established the existence results of the functional equations in game theory using dynamic programming method. In this paper an optimal inventory equation is considered which is concerned with the problem of stocking a supply of items to meet an uncertain demand under the assumptions of various costs associated with over supply and under supply when the total time period is unlimited and the existence of the optimal policy is established using fixed point theorem.

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2 Formulation of optimal inventory equation

Suppose in an inventory management system certain set of items is to be ordered at various specified times. The cost of ordering depends upon the quantity ordered for each item. At various other times demands are made upon the stocks of these items. The incentive for ordering leads a penalty cost which is assessed when ever the demand for an item exceeds the supply. The following notations are used to formulate the model.
$S =$ The random variable representing the demand of an item. It is assumed that S is a continuous random variable, distributed over $[0, \infty)$, with density function $\phi(s)$. $\phi(s)ds$ is the probability that the demand will lie between s and $s + ds$. $\int_0^\infty \phi(s)ds = 1$.
$k(z) =$ The cost of ordering z items initially to increase the stock level.
$p(z) =$ The cost of ordering z items to meet an excess, z, of demand over supply, i.e the penalty cost.
It is assumed that there are N stages and the time over which this process operates is unbounded that is the above functions are independent of time. Let x denotes the stock level at the beginning of the process. Suppose $y_n = y_n(x)$ specify that the quantity y_n is to be ordered at n^{th} stage when the stock level is x. The set of functions $\{y_1, y_2, ...y_N\}$ is called a policy. Each policy is associated with a certain expected total cost over this N-stage process, involving initial ordering and penalty costs. The objective is to determine the ordering policy at each stage which will minimize some average function of the overall costs of the process when N is very large. At each stage of the process, the problem is associated with the state variable, x (the supply of stock) and the number of remaining stages. Since there is no restriction over time, a discount factor should be introduced to prevent infinite costs from entering. Let $0 < a < 1$ be the fixed discount ratio for each period. Suppose
$f_n(x) =$Expected total cost for an n stage process starting with an initial policy x and using an optimal ordering policy.
So, if a quantity is ordered up to a certain level y, and y is to be chosen to minimize the expected cost, we see that

$$f_1(x) = \begin{matrix} inf \\ y > x \end{matrix} \left[k(y - x) + a \int_y^\infty p(s - y)\phi(s)ds \right]$$

in general for $n \geq 2$, we have the following recurrence relation.

$$\begin{aligned} f_n(x) = \begin{matrix} inf \\ y > x \end{matrix} \big[k(y - x) \quad &+ \quad a \int_y^\infty p(s - y)\phi(s)ds \\ &+ \quad a f_{n-1}(0) \int_y^\infty \phi(s)ds + a \int_0^y f_{n-1}(y - s)\phi(s)ds \big] \end{aligned}$$

If the number of stages is very large solution of this infinite process can be used as an approximation to the solution of the finite. In that case we may set,

$f(x)$ as expected total discount cost starting with an initial supply x. Then in place of the above equation, we obtain

$$f(x) = \inf_{y > x} \Big[k(y-x) \quad + \quad a\int_y^\infty p(s-y)\phi(s)ds \quad + \quad af(0)\int_y^\infty \phi(s)ds + a\int_0^y f(y-s)\phi(s)ds\Big] \quad (1)$$

To prove the existence of solution of the above optimal equation, we will use the following lemma which is the extension of Brauer fixed point theorem[1].

LEMMA 2.1 *Let $\langle E, d\rangle$ be a complete metric space and A be a mapping of E into itself. Suppose for any $x, y \in E$, $d(Ax, Ay) \leq \rho(d(x, y))$ where $\rho : [0, \infty) \to [0, \infty)$ is non decreasing, continuous on right and $\rho(r) < r$ for $r > 0$ and for every $x \in E$, there is a positive number λ_x such that $d(x, A^n(x)) \leq \lambda_x \forall n$. Then A has a unique fixed point.*

3 Existence of the solution of equation (1)

Throughout this paper Q denotes the state space constituting the state variable x and D denotes the decision space constituting the decision variable y. $B(Q)$ denotes the metric space of all real valued bounded functions on Q, with the usual "sup" norm as follows. For $\psi_1, \psi_2 \in Q$,

$$d(\psi_1, \psi_2) = \sup_{x \in Q} \mid \psi_1(x) - \psi_2(x) \mid$$

Then d is a metric and $\langle B(Q), d\rangle$ is a complete metric space. Let T be an operator on $R \times Q \times B(Q)$ defined by

$$T(y, x, f) = k(y-x) + a\Big[\int_y^\infty p(s-y)\phi(s)ds + f(0)\int_y^\infty \phi(s)ds + \int_0^y f(y-s)\phi(s)ds\Big]$$

Then equation (1) becomes $f(x) = \inf_{y > x} T(y, x, f)$.

Theorem 3.1 *The optimal inventory equation (1) possesses a unique bounded solution if $T(x, y, z_1) - T(x, y, z_2) \leq \rho(\mid z_1(x) - z_2(x) \mid)$, where $\rho : [0, \infty) \to [0, \infty)$ is a non decreasing continuous function on the right and $\rho(r) < r$ for $r > 0$.*

Proof: Let $A : B(Q) \to B(Q)$ be defined by $A(t) = \psi$ for $t \in B(Q)$, where

$$\psi(x) = \inf_{y > x} T(y, x, t)$$

Hence any fixed point of A is a bounded solution of (1) and any bounded solution of (1) is a fixed point of A. For $i = 1,2$ and $x \in Q$, $\psi_i(x) = \underset{y>x}{inf}\ T(y,x,t_i)$. Using the property of infimum it can be concluded that $\mid \psi_1(x) - \psi_2(x) \mid \leq \rho(\mid t_1(x) - t_2(x) \mid)$. This implies $d(At_1, At_2) \leq \rho(d(t_1,t_2))$, which is the first condition of the lemma. Again if we set $t_n = A^n(t)$ for $t \in B(Q)$, n=1,2,..., where

$$t(x) = \underset{y>x}{inf}\ \big[k(y-x) \ + \ a(\int_y^\infty p(s-y)\phi(s)ds \\ + \ t(0)\int_y^\infty \phi(s)ds + \int_0^y t(y-s)\phi(s)ds)\big]$$

then $t_n(x) = A^n t(x) = A^{n-1}At(x) = A^{n-1}\psi(x) = \psi_{n-1}(x) = \underset{y>x}{inf}\ T(y,x,t_{n-1})$

Thus for any $y \geq x$, we have $t(x) - t_n(x) \leq \underset{y>x}{sup}\ [T(y,x,t) - T(y,x,t_{n-1})]$. Let the supremum exists at y_0. Then

$$\mid t(x) - t_n(x) \mid \leq a\left[\mid t(0) - t_{n-1}(0) \mid . \mid \int_y^\infty \phi(s)ds \mid . \mid \int_0^{y_0} (t(y_0 - s) - t_{n-1}(y_0 - s))\phi(s)ds \mid\right]$$

Since $t \in B(Q)$, there exists real numbers λ_1 and λ_2 such that$\mid t(0) - t_{n-1}(0) \mid < \lambda_1$ and $\mid t(y_0 - s) - t_{n-1}(y_0 - s) \mid < \lambda_2$. If we set $\lambda_3 = \text{Max}(\lambda_1, \lambda_2)$,($\lambda_3$ depends upon t) then

$$\mid t(x) - t_n(x) \mid < a\lambda_3\big(\mid \int_{y_0}^\infty \phi(s)ds \mid + \mid \int_0^{y_0} \phi(s)ds \mid\big) \leq a\lambda_3 = \lambda_t \text{ (say)}$$

Hence $d(t, A^n(t) < \lambda_t$, which is the second condition of the lemma. Hence A has a unique fixed point i.e equation (1) has a unique bounded solution.

Theorem 3.2 *Under the following conditions the optimal inventory equation (1) possesses a solution.*

(i) $\int_0^\infty p(s)\phi(s)ds < \infty$, *$p$ is a monotonic increasing positive function with* $\int_0^\infty p(s)ds < \infty$

(ii) *k is bounded.*

(iii) $T(x,y,z_1) - T(x,y,z_2) \geq z_1 - z_2 \forall(x,y,z) \in R \times Q \times B(Q)$

Proof. Let a sequence of functions $\{f_n\}$ on Q is defined as follows. For $x \in Q$,

$$f_1(x) = \underset{y>x}{inf}\ k(y-x) + a\int_y^\infty p(s-y)\phi(s)ds$$

and for $n = 1, 2, ..., f_n(x) = \underset{y>x}{inf}\ T(y, x, f_{n-1})$. So

$$\begin{aligned} f_2(x) &= \underset{y>x}{inf}\ T(y, x, f_1) \\ &\geq \underset{y>x}{inf}\ [k(y-x) + a\int_y^\infty p(s-y)\phi(s)ds] \\ &+ \underset{y>x}{inf}\ [af_1(0)\int_y^\infty \phi(s)ds + a\int_0^y f_1(y-s)\phi(s)ds] \\ &\geq f_1(x) \end{aligned}$$

Assume that $f_k(x) \geq f_{k-1}(x), \forall x \in Q$. Then

$$f_{k+1}(x) - f_k(x) = \underset{y>x}{inf}\ T(y, x, f_k) - \underset{y>x}{inf}\ T(y, x, f_{k-1})$$

Let for some $y_1, y_2 > x$, $\underset{y>x}{inf}\ T(y, x, f_k) = T(y_1, x, f_k)$ and $\underset{y>x}{inf}\ T(y, x, f_{k-1}) = T(y_2, x, f_{k-1})$. Then $T(y_1, x, f_{k-1}) \geq T(y_2, x, f_{k-1})$. Hence by (iii),

$$\begin{aligned} f_{k+1}(x) - f_k(x) &\geq T(y_1, x, f_k) - T(y_1, x, f_{k-1}) + T(y_1, x, f_{k-1}) - T(y_2, x, f_{k-1}) \\ &\geq T(y_1, x, f_k) - T(y_1, x, f_{k-1}) > f_k(x) - f_{k-1}(x) \geq 0 \end{aligned}$$

Hence $\{f_n(x)\}$ is a monotonically increasing sequence. It remains to prove the boundedness of $\{f_n(x)\}$. Since $\int_0^\infty p(s)\phi(s)ds < \infty$ and p is a monotonically increasing positive function, k is bounded, we can choose a constant C such that

$$\underset{y>x}{sup}\ k(y-x) + a\int_y^\infty p(s-y)\phi(s)ds < C$$

Hence $f_1(0) < C$ and $f_1(y-s) < C$. So

$$\begin{aligned} f_2(x) &\leq \underset{y>x}{sup}\ [k(y-x) + a\int_0^\infty p(s-y)\phi(s)ds] \\ &+ \underset{y>x}{inf}\ [af_1(0)\int_y^\infty \phi(s)ds + a\int_0^y f_1(y-s)\phi(s)ds] \end{aligned}$$

So for any $y > x$, $f_2(x) < C + aC[\int_0^\infty \phi(s)ds] = C + aC$

$$\begin{aligned} f_3(x) &\leq \underset{y>x}{sup}\ [k(y-x) + a\int_0^\infty p(s-y)\phi(s)ds] \\ &+ \underset{y>x}{inf}\ [af_2(0)\int_y^\infty \phi(s)ds + a\int_0^y f_2(y-s)\phi(s)ds] \end{aligned}$$

Using the above inequalities, for any $y > x$, $f_3(x) < C + aC + a^2C$ and so on. Thus

$$f_n(x) < C + aC + a^2C + ... + a^{n-1}C = \frac{1-a^n}{1-a}C$$

Hence the sequence $\{f_n(x)\}$ is monotonically increasing and bounded above. i.e $\{f_n(x)\}$ is convergent. Let $\lim_{n \to \infty} f_n(x) = \bar{f}(x)$. Hence it can be concluded that $\lim_{n \to \infty} T(y, x, f_n) = \bar{f}(x)$. So $\bar{f}$ is a solution of equation(1).

4 Conclusion

The method of successive approximation not only determines the existence of optimal policy but also determines it's behavior such as uniqueness, boundedness etc. This infinite process in inventory system and the existence of optimal policy has some significant application in the sense that if an inventory management team of a large scale industry can study the inventory system of the previous stages and determine the recurrence relation then it is possible to guess the behavior of the optimal policy in future (after long years) under similar circumstances. Existence of the optimal policy of the multistage allocation process of other types of inventory models in different situations i.e with penalty cost, time lag, constant stock level etc can also be derived under similar type assumptions, which is the further scope of this paper.

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