

# Optimal Sequential Testing of Two Simple Hypotheses in Presence of Control Variables

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## Abstract

Suppose that at any stage of a statistical experiment a control variable  $X$  that affects the distribution of the observed data  $Y$  can be used. The distribution of  $Y$  depends on some unknown parameter  $\theta$ , and we consider the classical problem of testing a simple hypothesis  $H_0 : \theta = \theta_0$  against a simple alternative  $H_1 : \theta = \theta_1$  allowing the data to be controlled by  $X$ , in the following sequential context.

The experiment starts with assigning a value  $X_1$  to the control variable and observing  $Y_1$  as a response. After some analysis, we choose another value  $X_2$  for the control variable, and observe  $Y_2$  as a response, etc. It is supposed that the experiment eventually stops, and at that moment a final decision in favour of  $H_0$  or  $H_1$  is to be taken.

In this article, our aim is to characterize the structure of optimal sequential procedures, based on this type of data, for testing a simple hypothesis against a simple alternative.

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## 1 Introduction. Problem Set-Up.

Let us suppose that at any stage of a statistical experiment a "control variable"  $X$  that affects the distribution of the observed data  $Y$  can be used. "Statistical" means that the distribution of  $Y$  depends on some unknown parameter  $\theta$ ,

and we have the usual goal of statistical analysis: to obtain some information about the true value of  $\theta$ . In this work, we consider the classical problem of testing a simple hypothesis  $H_0 : \theta = \theta_0$  versus a simple alternative  $H_1 : \theta = \theta_1$  allowing the data to be controlled by  $X$ , in the following "sequential" context.

The experiment starts with assigning a value  $X_1$  to the control variable and observing  $Y_1$  as a response. After some analysis, we choose another value  $X_2$  for the control variable, and observe  $Y_2$  as a response. Analyzing this, we choose  $X_3$  for the third stage, get  $Y_3$ , and so on. In this way, we obtain a sequence  $X_1, \dots, X_n, Y_1, \dots, Y_n$  of experimental data,  $n = 1, 2, \dots$ . It is supposed that the experiment eventually stops, and at that moment a final decision in favour of  $H_0$  or  $H_1$  is to be taken.

In this article, our aim is to characterize the structure of optimal sequential procedures, based on this type of data, for testing a simple hypothesis against a simple alternative.

Let us write, briefly,  $X^{(n)}$  instead of  $(X_1, \dots, X_n)$ ,  $Y^{(n)}$  instead of  $(Y_1, \dots, Y_n)$ , etc. Let us define a (randomized) sequential hypothesis testing procedure as a triplet  $(\chi, \psi, \phi)$  of a *control policy*  $\chi$ , a *stopping rule*  $\psi$ , and a *decision rule*  $\phi$ , with

$$\begin{aligned}\chi &= (\chi_1, \chi_2, \dots, \chi_n, \dots), \\ \psi &= (\psi_1, \psi_2, \dots, \psi_n, \dots), \\ \phi &= (\phi_1, \phi_2, \dots, \phi_n, \dots),\end{aligned}$$

where

$$\chi_n = \chi_n(x^{(n-1)}, y^{(n-1)})$$

$n = 1, 2, \dots$  are supposed to be measurable functions with values in the space of values of the control variable, and the functions

$$\psi_n = \psi_n(x^{(n)}, y^{(n)}), \quad \phi_n = \phi_n(x^{(n)}, y^{(n)})$$

are supposed to be some measurable functions with values in  $[0, 1]$ .

The interpretation of these functions is as follows.

The experiments starts at stage  $n = 1$  applying  $\chi_1$  to determine the initial control  $x_1$ . Using this control, the first data  $y_1$  is observed.

At any stage  $n \geq 1$ : the value of  $\psi_n(x^{(n)}, y^{(n)})$  is interpreted as the conditional probability *to stop and proceed to decision making*, given that that we came to that stage and that the observations were  $(y_1, y_2, \dots, y_n)$  after the respective controls  $(x_1, x_2, \dots, x_n)$  have been applied. If there is no stop, the experiments continues to the next stage, defining first the new control value  $x_{n+1}$  by applying the control policy:  $x_{n+1} = \chi_{n+1}(x^{(n)}, y^{(n)})$  and then taking an additional observation  $y_{n+1}$  using control  $x_{n+1}$ .

Then the rule  $\psi_{n+1}$  is applied to  $(x_1, \dots, x_{n+1}; y_1, \dots, y_{n+1})$  in the same way as as above, etc., until the experiment eventually stops.

It is supposed that when the experiment stops, a decision *to accept or to reject*  $H_0$  is to be made. The function  $\phi_n(x^{(n)}, y^{(n)})$  is interpreted as the conditional probability *to reject* the null-hypothesis  $H_0$ , given that the experiment stops at stage  $n$  being  $(y_1, \dots, y_n)$  the data vector observed and  $(x_1, \dots, x_n)$  the respective controls applied.

The control policy  $\chi$  generates, by the above process, a sequence of random variables  $X_1, X_2, \dots, X_n$ , recursively by

$$X_{n+1} = \chi_{n+1}(X^{(n)}, Y^{(n)}).$$

The stopping rule  $\psi$  generates, by the above process, a random variable  $\tau_\psi$  (*stopping time*) whose distribution is given by

$$P_\theta^X(\tau_\psi = n) = E_\theta^X(1 - \psi_1)(1 - \psi_2) \dots (1 - \psi_{n-1})\psi_n. \quad (1)$$

Here, and throughout the paper, we interchangeably use  $\psi_n$  both for

$$\psi_n(x^{(n)}, y^{(n)})$$

and for

$$\psi_n(X^{(n)}, Y^{(n)}),$$

and so do we for any other function

$$F_n = F_n(x^{(n)}, y^{(n)}).$$

This does not cause any problem if we adopt the following agreement: when  $F_n$  is under probability or expectation sign, it is  $F_n(X^{(n)}, Y^{(n)})$ , otherwise it is  $F_n(x^{(n)}, y^{(n)})$ .

For a sequential testing procedure  $(\chi, \psi, \phi)$  let us define *the type I error probability* as

$$\alpha(\chi, \psi, \phi) = P_{\theta_0}(\text{reject } H_0) = \sum_{n=1}^{\infty} E_{\theta_0}^X(1 - \psi_1) \dots (1 - \psi_{n-1})\psi_n\phi_n \quad (2)$$

and *the type II error probability* as

$$\beta(\chi, \psi, \phi) = P_{\theta_1}(\text{accept } H_0) = \sum_{n=1}^{\infty} E_{\theta_1}^X(1 - \psi_1) \dots (1 - \psi_{n-1})\psi_n(1 - \phi_n). \quad (3)$$

Normally, we would like to keep them below some specified levels:

$$\alpha(\chi, \psi, \phi) \leq \alpha \quad (4)$$

and

$$\beta(\chi, \psi, \phi) \leq \beta \quad (5)$$

with some  $\alpha, \beta \in (0, 1)$ .

Another important characteristic of a sequential testing procedure is the *average sample number*:

$$N(\theta; \chi, \psi) = E_{\theta}^{\chi} \tau_{\psi} = \begin{cases} \sum_{n=1}^{\infty} n P_{\theta}^{\chi}(\tau_{\psi} = n), & \text{if } P_{\theta}^{\chi}(\tau_{\psi} < \infty) = 1, \\ \infty & \text{otherwise.} \end{cases} \quad (6)$$

Our main goal is minimizing  $N(\chi, \psi) = N(\theta_0; \chi, \psi)$  over all sequential testing procedures subject to (4) and (5). Our method is essentially the same that we used in [3] in the problem of sequential testing of two simple hypotheses without control variables.

In Section 2, we reduce the problem of minimizing  $N(\chi, \psi)$  under constraints (4) and (5) to an unconstrained minimization problem. The new objective function is the Lagrange-multiplier function  $L(\chi, \psi, \phi)$ .

In Section 3, we find

$$L(\chi, \psi) = \inf_{\phi} L(\chi, \psi, \phi).$$

In Section 4, we minimize  $L(\chi, \psi)$  in the class of truncated stopping rules, i.e. such that  $\psi_N \equiv 1$ .

In Section 5, we characterize the structure of optimal strategy  $(\chi, \psi)$  in the class of non-truncated stopping rules.

In Section 6, the likelihood ratio structure for optimal strategy is given.

In Section 7, we apply the results obtained in Section 2 – Section 5 to minimizing the average sample number  $N(\chi, \psi)$  over all sequential testing procedures subject to (4) and (5).

## 2 Reduction to Non-Constrained Minimization

To proceed with minimizing (6) over the testing procedures subject to (4) and (5) let us define the following Lagrange-multiplier function:

$$L(\chi, \psi, \phi) = N(\chi, \psi) + \lambda_0 \alpha(\chi, \psi, \phi) + \lambda_1 \beta(\chi, \psi, \phi) \quad (7)$$

where  $\lambda_0 \geq 0$  and  $\lambda_1 \geq 0$  are some constant multipliers.

Let  $\Delta$  be a class of sequential testing procedures.

The usual relation between the constrained and the non-constrained minimization is given by the following

**Theorem 2.1.** *Let exist  $\lambda_0 > 0$  and  $\lambda_1 > 0$  and a testing procedure  $(\chi^*, \psi^*, \phi^*) \in \Delta$  such that for any other testing procedure  $(\chi, \psi, \phi) \in \Delta$*

$$L(\chi^*, \psi^*, \phi^*) \leq L(\chi, \psi, \phi) \quad (8)$$

holds and such that

$$\alpha(\chi^*, \psi^*, \phi^*) = \alpha \quad \text{and} \quad \beta(\chi^*, \psi^*, \phi^*) = \beta. \quad (9)$$

Then for any testing procedure  $(\chi, \psi, \phi) \in \Delta$  satisfying

$$\alpha(\chi, \psi, \delta) \leq \alpha \quad \text{and} \quad \beta(\chi, \psi, \delta) \leq \beta \quad (10)$$

it holds

$$N(\chi^*, \psi^*) \leq N(\chi, \psi). \quad (11)$$

The inequality in (11) is strict if at least one of the equalities (10) is strict.

*Proof.* Let  $(\chi, \psi, \phi) \in \Delta$  be any testing procedure satisfying (10). Because of (8):

$$\begin{aligned} L(\chi^*, \psi^*, \phi^*) &= N(\chi^*, \psi^*) + \lambda_0 \alpha(\chi^*, \psi^*, \phi^*) + \lambda_1 \beta(\chi^*, \psi^*, \phi^*) \\ &\leq L(\chi, \psi, \phi) = N(\chi, \psi) + \lambda_0 \alpha(\chi, \psi, \phi) + \lambda_1 \beta(\chi, \psi, \phi) \end{aligned} \quad (12)$$

$$\leq N(\chi, \psi) + \lambda_0 \alpha + \lambda_1 \beta, \quad (13)$$

where to get the last inequality we used (4) and (5).

So,

$$N(\chi^*, \psi^*) + \lambda_0 \alpha(\chi^*, \psi^*, \phi^*) + \lambda_1 \beta(\chi^*, \psi^*, \phi^*) \leq N(\chi, \psi) + \lambda_0 \alpha + \lambda_1 \beta,$$

and taking into account conditions (9) we get from this that

$$N(\chi^*, \psi^*) \leq N(\chi, \psi).$$

The get the last statement of the theorem we note that if  $N(\chi^*, \psi^*) = N(\chi, \psi)$  then there are equalities in (12)-(13) instead of inequalities which is only possible if  $\alpha(\chi, \psi, \phi) = \alpha$  and  $\beta(\chi, \psi, \phi) = \beta$ .  $\square$

### 3 Optimal Decision Rules

In this section, we start solving the problem of minimizing the Lagrange-multiplier function  $L(\chi, \psi, \phi)$  over all sequential testing procedures: we first find

$$\inf_{\phi} L(\chi, \psi, \phi),$$

and the corresponding decision rule, at which this infimum is attained.

Let  $I_A$  be the indicator function of the event  $A$ .

From this time on, we suppose that for any  $n = 1, 2, \dots$ , the random variable  $Y$ , when a control  $x$  is applied, has a probability "density" function

$$f_\theta(y|x) \tag{14}$$

(Radon-Nicodym derivative of its distribution) with respect to a  $\sigma$ -finite measure  $\mu$  on the respective space. We are supposing as well that, at any stage  $n \geq 1$ , given control values  $x_1, x_2, \dots, x_n$  applied, the observations  $Y_1, Y_2, \dots, Y_n$  are independent, i.e. their joint probability density function, conditionally on given controls  $x_1, x_2, \dots, x_n$ , can be calculated as

$$f_\theta^n(x_1, \dots, x_n; y_1, \dots, y_n) = \prod_{i=1}^n f_\theta(y_i|x_i), \tag{15}$$

with respect to the product-measure  $\mu^n = \mu \otimes \dots \otimes \mu$  of  $\mu$   $n$  times by itself. It is easy to see that any expectation, which uses a control policy  $\chi$ , can be expressed as

$$E_\theta^\chi g(Y^{(n)}) = \int g(y^{(n)}) f_\theta^{n,\chi}(y^{(n)}) d\mu^n(y^{(n)}),$$

where

$$f_\theta^{n,\chi}(y^{(n)}) = \prod_{i=1}^n f_\theta(y_i|x_i)$$

with

$$x_i = \chi_i(x^{(i-1)}, y^{(i-1)}) \tag{16}$$

for any  $i = 1, 2, \dots$ .

Similarly, for any function  $F_n = F_n(x^{(n)}, y^{(n)})$  let us define

$$F_n^\chi(y^{(n)}) = F_n(x^{(n)}, y^{(n)})$$

where  $x_1, \dots, x_n$  are defined by (16).

As a first step of minimization of  $L(\chi, \psi, \phi)$ , let us prove the following

**Theorem 3.1.** *For any  $\lambda_0 \geq 0$  and  $\lambda_1 \geq 0$  and for any sequential testing procedure  $(\chi, \psi, \phi)$*

$$L(\chi, \psi, \phi) \geq L(\chi, \psi, \phi^*) \tag{17}$$

$$= N(\chi, \psi) + \sum_{n=1}^{\infty} \int (1 - \psi_1^\chi) \dots (1 - \psi_{n-1}^\chi) \psi_n^\chi \min\{\lambda_0 f_{\theta_0}^{n,\chi}, \lambda_1 f_{\theta_1}^{n,\chi}\} d\mu^n. \tag{18}$$

with

$$\phi^* = (\phi_1^*, \phi_2^*, \dots, \phi_n^*, \dots) \tag{19}$$

where

$$\phi_n^* = I_{\{\lambda_0 f_{\theta_0}^n \leq \lambda_1 f_{\theta_1}^n\}} \tag{20}$$

*Proof.* Inequality (17) is equivalent to

$$\lambda_0\alpha(\chi, \psi, \phi) + \lambda_1\beta(\chi, \psi, \phi) \geq \lambda_0\alpha(\chi, \psi, \phi^*) + \lambda_1\beta(\chi, \psi, \phi^*). \quad (21)$$

We prove (21) by finding a lower bound for the left-hand side of (21) and proving that this lower bound is attained at  $\phi = \phi^*$  defined by (20).

To do this, we will use the following simple

**Lemma 3.2.** [3] *Let  $\phi, F_1, F_2$  be some measurable functions on a measurable space with a measure  $\mu$ , such that*

$$0 \leq \phi(x) \leq 1, \quad F_1(x) \geq 0, \quad F_2(x) \geq 0,$$

and

$$\int \min\{F_1(x), F_2(x)\}d\mu(x) < \infty.$$

Then

$$\begin{aligned} \int (\phi(x)F_1(x) + (1 - \phi(x))F_2(x))d\mu(x) \\ \geq \int \min\{F_1(x), F_2(x)\}d\mu(x) \end{aligned} \quad (22)$$

with an equality if and only if

$$I_{\{F_1(x) < F_2(x)\}} \leq \phi(x) \leq I_{\{F_1(x) \leq F_2(x)\}} \quad (23)$$

$\mu$ -almost everywhere.

Starting with the proof of (21), let us give to the left-hand side of it the form

$$\begin{aligned} &\lambda_0\alpha(\chi, \psi, \phi) + \lambda_1\beta(\chi, \psi, \phi) \\ &= \sum_{n=1}^{\infty} \int (1 - \psi_1^x) \dots (1 - \psi_{n-1}^x) \psi_n^x [\phi_n^x \lambda_0 f_{\theta_0}^{n,x} + (1 - \phi_n^x) \lambda_1 f_{\theta_1}^{n,x}] d\mu^n \end{aligned} \quad (24)$$

(see (2)).

Applying Lemma 1 to each summand in (24) we immediately have:

$$\begin{aligned} &\lambda_0\alpha(\chi, \psi, \phi) + \lambda_1\beta(\chi, \psi, \phi) \\ &\geq \sum_{n=1}^{\infty} \int (1 - \psi_1^x) \dots (1 - \psi_{n-1}^x) \psi_n^x \min\{\lambda_0 f_{\theta_0}^{n,x}, \lambda_1 f_{\theta_1}^{n,x}\} d\mu^n \end{aligned} \quad (25)$$

with an equality if

$$\phi_n = I_{\{\lambda_0 f_{\theta_0}^n \leq \lambda_1 f_{\theta_1}^n\}} = \phi_n^*$$

for any  $n = 1, 2, \dots$ . But in this case the right-hand side of (25) is  $\lambda_0\alpha(\psi, \phi^*) + \lambda_1\beta(\psi, \phi^*)$ , so we get (21).  $\square$

**Remark 3.3.** *It is easy to see, using (6) and (25), that for any  $(\chi, \psi)$  such that  $P_{\theta_0}^\chi(\tau_\psi < \infty)$  the minimum value  $L(\chi, \psi, \phi^*)$  in (17) can be represented as*

$$L(\chi, \psi) = \sum_{n=1}^{\infty} \int (1 - \psi_1^\chi) \dots (1 - \psi_{n-1}^\chi) \psi_n^\chi (n f_{\theta_0}^{n,\chi} + l_n^\chi) d\mu^n, \tag{26}$$

where, by definition,

$$l_n = \min\{\lambda_0 f_{\theta_0}^n, \lambda_1 f_{\theta_1}^n\}.$$

Let us denote, for the rest of this article,

$$s_n^\psi = (1 - \psi_1) \dots (1 - \psi_{n-1}) \psi_n \quad \text{and} \quad c_n^\psi = (1 - \psi_1) \dots (1 - \psi_{n-1})$$

for any  $n = 1, 2, \dots$ . Respectively,

$$s_n^{\psi,\chi} = (1 - \psi_1^\chi) \dots (1 - \psi_{n-1}^\chi) \psi_n^\chi \quad \text{and} \quad c_n^{\psi,\chi} = (1 - \psi_1^\chi) \dots (1 - \psi_{n-1}^\chi)$$

for any  $n = 1, 2, \dots$ .

Let also

$$C_n^{\psi,\chi} = \{y^{(n)} : (1 - \psi_1^\chi(y^{(1)})) \dots (1 - \psi_{n-1}^\chi(y^{(n-1)})) > 0\},$$

for any  $n \geq 2$ , and let  $C_1^{\psi,\chi}$  be the space of all  $y^{(1)}$ , and finally let

$$\bar{C}_n^{\psi,\chi} = \{y^{(n)} : (1 - \psi_1^\chi(y^{(1)})) \dots (1 - \psi_n^\chi(y^{(n)})) > 0\},$$

for any  $n \geq 1$ .

### 4 Truncated Stopping Rules

Our next goal is to find a control policy  $\chi$  and a stopping rule  $\psi$  minimizing the value of  $L(\chi, \psi)$  in (26).

In this section, we solve, as an intermediate step, the problem of minimization of  $L(\chi, \psi)$  over all  $\chi$  and  $\psi$ , where  $\psi \in \Delta^N$ , the class of truncated stopping rules, that is,

$$\psi = (\psi_1, \psi_2, \dots, \psi_{N-1}, 1, \dots). \tag{27}$$

The following lemma takes over a large part of work of doing this.

**Lemma 4.1.** *Let  $r \geq 2$  be any natural number, and let  $v_r = v_r(x^{(r)}, y^{(r)})$  be any measurable function. Then*

$$\sum_{n=1}^{r-1} \int s_n^{\psi,\chi} (n f_{\theta}^{n,\chi} + l_n^\chi) d\mu^n + \int c_r^{\psi,\chi} (r f_{\theta}^{r,\chi} + v_r^\chi) d\mu^r$$

$$\geq \sum_{n=1}^{r-2} \int s_n^{\psi, \chi} (n f_{\theta}^{n, \chi} + l_n^{\chi}) d\mu^n + \int c_{r-1}^{\psi, \chi} ((r-1) f_{\theta}^{r-1, \chi} + v_{r-1}^{\chi}) d\mu^{r-1}, \tag{28}$$

with

$$v_{r-1} = \min\{l_{r-1}, f_{\theta}^{r-1} + R_{r-1}\}, \tag{29}$$

where

$$R_{r-1}(x^{(r-1)}, y^{(r-1)}) = \min_{x_r} \int v_r(x_1, \dots, x_r; y_1, \dots, y_r) d\mu(y_r) \tag{30}$$

There is an equality in (28) if and only if

$$I_{\{l_{r-1}^{\chi} < f_{\theta}^{r-1, \chi} + R_{r-1}^{\chi}\}} \leq \psi_{r-1}^{\chi} \leq I_{\{l_{r-1}^{\chi} \leq f_{\theta}^{r-1, \chi} + R_{r-1}^{\chi}\}} \tag{31}$$

$\mu^{r-1}$ -almost everywhere on  $C_{r-1}^{\psi, \chi}$ , and

$$\int v_r^{\chi}(y^{(r)}) d\mu(y_r) = R_{r-1}^{\chi}(y^{(r-1)}) \tag{32}$$

$\mu^{r-1}$ -almost everywhere on  $\bar{C}_{r-1}^{\psi, \chi}$ . (We suppose that  $R_{r-1}$  defined by (30) is a measurable function of its arguments).

*Proof.* To prove (28), it is sufficient to show that

$$\begin{aligned} & \int s_{r-1}^{\psi, \chi} ((r-1) f_{\theta}^{r-1, \chi} + l_{r-1}^{\chi}) d\mu^{r-1} + \int c_r^{\psi, \chi} (r f_{\theta}^{r, \chi} + v_r^{\chi}) d\mu^r \\ & \geq \int c_{r-1}^{\psi, \chi} ((r-1) f_{\theta}^{r-1, \chi} + v_{r-1}^{\chi}) d\mu^{r-1}. \end{aligned} \tag{33}$$

By the Fubini theorem, the left-hand side of (33) is equal to

$$\begin{aligned} & \int s_{r-1}^{\psi, \chi} ((r-1) f_{\theta}^{r-1, \chi} + l_{r-1}^{\chi}) d\mu^{r-1} + \int c_r^{\psi, \chi} \left( \int (r f_{\theta}^{r, \chi} + v_r^{\chi}) d\mu(y_r) \right) d\mu^{r-1} \\ & = \int c_{r-1}^{\psi, \chi} [\psi_{r-1}^{\chi} ((r-1) f_{\theta}^{r-1, \chi} + l_{r-1}^{\chi}) + (1 - \psi_{r-1}^{\chi}) \int (r f_{\theta}^{r, \chi} + v_r^{\chi}) d\mu(y_r)] d\mu^{r-1}. \end{aligned} \tag{34}$$

Because of (15),

$$\int f_{\theta}^r(x^{(r)}, y^{(r)}) d\mu(y_r) = f_{\theta}^{r-1}(x^{(r-1)}, y^{(r-1)}),$$

so that the right-hand side of (34) transforms to

$$\begin{aligned} & \int c_{r-1}^{\psi, \chi} \left[ (r-1)f_{\theta}^{r-1, \chi} + \psi_{r-1}^{\chi} l_{r-1}^{\chi} + (1 - \psi_{r-1}^{\chi})(f_{\theta}^{r-1, \chi} + \int v_r^{\chi} d\mu(y_r)) \right] d\mu^{r-1} \\ & \geq \int c_{r-1}^{\psi, \chi} [(r-1)f_{\theta}^{r-1, \chi} + \psi_{r-1}^{\chi} l_{r-1}^{\chi} + (1 - \psi_{r-1}^{\chi})(f_{\theta}^{r-1, \chi} + R_{r-1}^{\chi})] d\mu^{r-1} \end{aligned} \quad (35)$$

Applying Lemma 3.2 with

$$\phi = \psi_{r-1}^{\chi}, \quad F_1 = c_{r-1}^{\psi, \chi} l_{r-1}^{\chi}, \quad F_2 = c_{r-1}^{\psi, \chi} (f_{\theta}^{r-1, \chi} + R_{r-1}^{\chi}),$$

we see that the right-hand side of (35) is greater than or equal to

$$\begin{aligned} & \int c_{r-1}^{\psi, \chi} [(r-1)f_{\theta}^{r-1, \chi} + \min\{l_{r-1}^{\chi}, f_{\theta}^{r-1, \chi} + R_{r-1}^{\chi}\}] d\mu^{r-1} \\ & = \int c_{r-1}^{\psi, \chi} [(r-1)f_{\theta}^{r-1, \chi} + v_{r-1}^{\chi}] d\mu^{r-1}, \end{aligned} \quad (36)$$

by the definition of  $v_{r-1}$  in (29).

Moreover, by the same Lemma 1, the right-hand side of (35) is equal to (36) if and only if  $\psi_{r-1}$  satisfies (31)  $\mu^{r-1}$ -almost everywhere on  $C_{r-1}^{\psi, \chi}$ .

In addition, there is an equality in (35) if and only if  $\chi_r$  satisfies (32)  $\mu^{r-1}$ -almost everywhere on  $\bar{C}_{r-1}^{\psi, \chi}$ .  $\square$

The following Theorem gives some lower bounds for  $L(\chi, \psi)$  when the stopping rule  $\psi$  is truncated ( $\psi \in \Delta^N$ ) and characterizes the stopping rules that attain these bounds.

**Theorem 4.2.** *Let  $\psi \in \Delta^N$  be any (truncated) stopping rule, and  $\chi$  any control policy. Then for any  $1 \leq r \leq N - 1$  the following inequalities hold true*

$$L(\chi, \psi) \geq \sum_{n=1}^r \int s_n^{\psi, \chi} (n f_{\theta_0}^{n, \chi} + l_n^{\chi}) d\mu^n + \int c_{r+1}^{\psi, \chi} \left( (r+1) f_{\theta_0}^{r+1, \chi} + V_{r+1}^{N, \chi} \right) d\mu^{r+1} \quad (37)$$

$$\geq \sum_{n=1}^{r-1} \int s_n^{\psi, \chi} (n f_{\theta_0}^{n, \chi} + l_n^{\chi}) d\mu^n + \int c_r^{\psi, \chi} \left( r f_{\theta_0}^{r, \chi} + V_r^{N, \chi} \right) d\mu^r, \quad (38)$$

where  $V_N^N \equiv l_N$ , and recursively for  $k = N, N - 1, \dots, 2$

$$V_{k-1}^N = \min\{l_{k-1}, f_{\theta_0}^{k-1} + R_{k-1}^N\}, \quad (39)$$

with

$$R_{k-1}^N = R_{k-1}^N(x^{(k-1)}; y^{(k-1)}) = \min_{x_k} \int V_k^N(x_1, \dots, x_k; y_1, \dots, y_k) d\mu(y_k). \quad (40)$$

The lower bound in (38) is attained if and only if

$$I_{\{x_k^X < f_{\theta_0}^{k,x} + R_k^{N,x}\}} \leq \psi_k^X \leq I_{\{x_k^X \leq f_{\theta_0}^{k,x} + R_k^{N,x}\}} \quad (41)$$

$\mu^k$ -almost everywhere on  $C_k^{\psi, X}$  and

$$R_k^{N,x}(y^{(k)}) = \int V_{k+1}^{N,x} d\mu(y_{k+1}) \quad (42)$$

$\mu^k$ -almost everywhere on  $\bar{C}_k^{\psi, X}$ , for any  $k = r, \dots, N - 1$ .

**Remark 4.3.** It is supposed in Theorem 4.2 and in what follows in this article that all the functions  $R_k^N$  defined by (40) are well-defined and measurable for any  $k = 1, 2, \dots, N$  and for any  $N = 1, 2, \dots$ , and that  $R_0^N$  defined by (44) below is well defined as well (this is true, for example, if  $x_i$  can take only a finite number of values for any  $i = 1, 2, \dots$ ).

*Proof.* There is an equality in (37) if  $r = N - 1$ . The rest of the proof immediately follows from Lemma 4.1 by induction.  $\square$

**Corollary 4.4.** For any truncated stopping rule  $\psi \in \Delta^N$ , and for any control rule  $\chi$

$$L(\chi, \psi) \geq 1 + R_0^N, \quad (43)$$

where

$$R_0^N = \min_{x_1} \int V_1^N(x_1; y_1) d\mu(y_1). \quad (44)$$

The lower bound in (43) is attained if and only if (41) is satisfied  $\mu^k$ -almost everywhere on  $C_k^{\psi, X}$  and (42) is satisfied  $\mu^k$ -almost everywhere on  $\bar{C}_k^{\psi, X}$ , for any  $k = 1, 2, \dots, N - 1$  and, additionally,

$$R_0^N = \int V_1^N(\chi_1; y_1) d\mu(y_1). \quad (45)$$

**Remark 4.5.** It is obvious that the testing procedure attaining the lower bound in (43) is optimal among all truncated testing procedures with  $\psi \in \Delta^N$ . But it only makes practical sense if

$$\min\{\lambda_0, \lambda_1\} > 1 + R_0^N.$$

The reason is that  $\min\{\lambda_0, \lambda_1\}$  can be considered as "the  $L(\chi, \psi)$ " function for a trivial sequential test  $(\psi_0, \phi_0)$  which, without taking any observations, makes the decision  $\phi_0 = I_{\{\lambda_0 \leq \lambda_1\}}$ . In this case there are no observations ( $N(\theta; \psi_0) = 0$ ) and it is easily seen that

$$L(\psi_0, \phi_0) = \lambda_0 \alpha(\psi_0, \phi_0) + \lambda_1 \beta(\psi_0, \phi_0) = \min\{\lambda_0, \lambda_1\}.$$

Thus, the inequality

$$\min\{\lambda_0, \lambda_1\} \leq 1 + R_0^N$$

means that the trivial test  $(\psi_0, \phi_0)$  is not worse than the best testing procedure with  $\psi$  from  $\Delta^N$ .

Because of that, we consider

$$V_0^N = \min\{\min\{\lambda_0, \lambda_1\}, 1 + R_0^N\}$$

as the minimum value of  $L(\chi, \psi)$  for  $\psi \in \Delta^N$ , when taking no observations is permitted. It is obvious that this is a particular case of (39) with  $k = 1$ , if we define  $l_0 \equiv \min\{\lambda_0, \lambda_1\}$  and  $f_{\theta_0}^0 \equiv 1$ .

### 5 Non-Truncated Stopping Rules

In this section we characterize the structure of general sequential testing procedures minimizing  $L(\chi, \psi)$ .

Let us define for any stopping rule  $\psi$  and any control policy  $\chi$

$$L_N(\chi, \psi) = \sum_{n=1}^{N-1} \int s_n^{\psi, \chi} (n f_{\theta_0}^{n, \chi} + l_n^{\chi}) d\mu^n + \int c_N^{\psi, \chi} (N f_{\theta_0}^{N, \chi} + l_N^{\chi}) d\mu^N. \quad (46)$$

This is the Lagrange-multiplier function corresponding to  $\psi$  truncated at  $N$ , i.e. the rule with the components  $\psi^N = (\psi_1, \psi_2, \dots, \psi_{N-1}, 1, \dots)$ :

$$L_N(\chi, \psi) = L(\chi, \psi^N).$$

Because  $\psi^N$  is truncated, the results of the preceding section apply, in particular, the inequalities of Theorem 4.2.

The idea of what follows is to make  $N \rightarrow \infty$ , to obtain some lower bounds for  $L(\chi, \psi)$  from (37) - (38).

And the first question is: what happens to  $L_N(\chi, \psi)$  when  $N \rightarrow \infty$ ?

Let us denote by  $F$  the set of all strategies  $(\chi, \psi)$  such that

$$\lim_{n \rightarrow \infty} E_{\theta_0}^{\chi} (1 - \psi_1) \dots (1 - \psi_n) = 0. \quad (47)$$

It is easy to see that (47) is equivalent to

$$P_{\theta_0}^{\chi} (\tau_{\psi} < \infty) = 1$$

(see (1)).

**Lemma 5.1.** For any strategy  $(\chi, \psi) \in \mathbb{F}$

$$\lim_{N \rightarrow \infty} L_N(\chi, \psi) = L(\chi, \psi).$$

*Proof.* Let  $L(\chi, \psi) < \infty$ , leaving the possibility  $L(\chi, \psi) = \infty$  till the end of the proof. Let us calculate the difference between  $L(\chi, \psi)$  and  $L_N(\chi, \psi)$  in order to show that it goes to zero as  $N \rightarrow \infty$ . By (46)

$$\begin{aligned} L(\psi) - L_N(\psi) &= \sum_{n=1}^{\infty} \int s_n^{\psi, \chi} (n f_{\theta_0}^{n, \chi} + l_n^{\chi}) d\mu^n \\ &\quad - \sum_{n=1}^{N-1} \int s_n^{\psi, \chi} (n f_{\theta_0}^{n, \chi} + l_n^{\chi}) d\mu^n - \int c_N^{\psi, \chi} (N f_{\theta_0}^{N, \chi} + l_N^{\chi}) d\mu^n \\ &= \sum_{n=N}^{\infty} \int s_n^{\psi, \chi} (n f_{\theta_0}^{n, \chi} + l_n^{\chi}) d\mu^n - \int c_N^{\psi, \chi} (N f_{\theta_0}^{N, \chi} + l_N^{\chi}) d\mu^n. \end{aligned} \tag{48}$$

The first summand on the right-hand side of (48) converges to zero, as  $N \rightarrow \infty$ , being the tail of a convergent series (this is because  $L(\chi, \psi) < \infty$ ).

We have further

$$\int c_N^{\psi, \chi} l_N^{\chi} d\mu^n \leq \lambda_0 \int c_N^{\psi, \chi} f_{\theta_0}^{N, \chi} d\mu^n = \lambda_0 E_{\theta_0}^{\chi} (1 - \psi_1) \dots (1 - \psi_{N-1}) \rightarrow 0$$

as  $N \rightarrow \infty$ , because of (47).

It remains to show that

$$\int c_N^{\psi, \chi} N f_{\theta_0}^{N, \chi} d\mu^n \rightarrow 0 \quad \text{as } N \rightarrow \infty. \tag{49}$$

But this is again due to the fact that  $L(\chi, \psi) < \infty$  which implies that

$$E_{\theta_0}^{\chi} \tau_{\psi} = \sum_{n=1}^{\infty} n P_{\theta_0}(\tau_{\psi} = n) < \infty.$$

Because this series is convergent,  $\sum_{n=N}^{\infty} n P_{\theta_0}^{\chi}(\tau_{\psi} = n) \rightarrow 0$ . Thus, using the Chebyshev inequality we have

$$N P_{\theta_0}^{\chi}(\tau_{\psi} \geq N) \leq E_{\theta_0}^{\chi} \tau_{\psi} I_{\{\tau_{\psi} \geq N\}} = \sum_{n=N}^{\infty} n P_{\theta_0}^{\chi}(\tau_{\psi} = n) \rightarrow 0$$

as  $N \rightarrow \infty$ , which completes the proof of (49).

Let now  $L(\chi, \psi) = \infty$ .

This means that

$$\sum_{n=1}^{\infty} \int s_n^{\psi, \chi} (n f_{\theta_0}^{n, \chi} + l_n^{\chi}) d\mu^n = \infty$$

which immediately implies by (46) that

$$L_N(\chi, \psi) \geq \sum_{n=1}^{N-1} \int s_n^{\psi, \chi} (n f_{\theta_0}^{n, \chi} + l_n^{\chi}) d\mu^n \rightarrow \infty.$$

□

The second question is about the behaviour of the functions  $V_r^N$  which participate in the inequalities of Theorem 4.2, as  $N \rightarrow \infty$ .

**Lemma 5.2.** *For any  $r \geq 1$  and for any  $N \geq r$*

$$V_r^N \geq V_r^{N+1}. \tag{50}$$

*Proof.* By induction over  $r = N, N - 1, \dots, 1$ .

Let  $r = N$ . Then by (39)

$$V_N^{N+1} = \min\{l_N, f_{\theta_0}^N + \min_{x_{N+1}} \int V_{N+1}^{N+1} d\mu(y_{N+1})\} \leq l_N = V_N^N.$$

If we suppose that (50) is satisfied for some  $r, N \geq r > 1$ , then

$$\begin{aligned} V_{r-1}^N &= \min\{l_{r-1}, f_{\theta_0}^{r-1} + \min_{x_r} \int V_r^N d\mu(y_r)\} \\ &\geq \min\{l_{r-1}, f_{\theta_0}^{r-1} + \min_{x_r} \int V_r^{N+1} d\mu(y_r)\} = V_{r-1}^{N+1}. \end{aligned}$$

Thus, (50) is satisfied for  $r - 1$  as well, which completes the induction. □

It follows from Lemma 5.2 that for any fixed  $r \geq 1$  the sequence  $V_r^N$  is non-increasing. So, there exists

$$V_r = \lim_{N \rightarrow \infty} V_r^N. \tag{51}$$

Now, everything is prepared for passing to the limit, as  $N \rightarrow \infty$ , in (37) and (38) with  $\psi = \psi^N$ .

**Theorem 5.3.** *Let  $\chi$  be any control policy and  $\psi$  any stopping rule. Then for any  $r \geq 1$  the following inequalities hold*

$$L(\chi, \psi) \geq \sum_{n=1}^r \int s_n^{\psi, \chi} (n f_{\theta_0}^{n, \chi} + l_n^X) d\mu^n + \int c_{r+1}^{\psi, \chi} ((r+1) f_{\theta_0}^{r+1, \chi} + V_{r+1}^X) d\mu^{r+1} \tag{52}$$

$$\geq \sum_{n=1}^{r-1} \int s_n^{\psi, \chi} (n f_{\theta_0}^{n, \chi} + l_n^X) d\mu^n + \int c_r^{\psi, \chi} (r f_{\theta_0}^{r, \chi} + V_r^X) d\mu^r, \tag{53}$$

where

$$V_r = \min\{l_r, f_{\theta_0}^r + R_r\}, \tag{54}$$

being

$$R_r = R_r(x^{(r)}, y^{(r)}) = \min_{x_{r+1}} \int V_{r+1}(x^{(r+1)}, y^{(r+1)}) d\mu(y_{r+1}). \tag{55}$$

In particular, for  $r = 1$ , the following lower bound holds true:

$$L(\chi, \psi) \geq 1 + \int V_1(\chi_1, y_1) d\mu(y_1) \geq 1 + R_0, \tag{56}$$

where, by definition,

$$R_0 = \min_{x_1} \int V_1(x_1, y_1) d\mu(y_1).$$

*Proof.* Let  $(\chi, \psi) \in F$  be any strategy. Then, by Lemma 5.1, the left-hand side of (37) tends to  $L(\chi, \psi)$  as  $N \rightarrow \infty$ .

By the Lebesgue monotone convergence theorem, in view of Lemma 5.2, passing to the limit on the right-hand sides of (37) and (38) is possible as well. Thus, (52) and (53) follow.

Let us now prove (54), starting from

$$V_r^N = \min\{l_r, f_{\theta_0}^r + R_r^N\}, \tag{57}$$

with

$$R_r^N = \min_{x_{r+1}} \int V_{r+1}^N d\mu(y_{r+1}) \tag{58}$$

(see (39) and (40), respectively).

By Lemma 5.2, the left-hand side of (57) tends to  $V_r$ . Additionally,

$$R_r^N = \min_{x_{r+1}} \int V_{r+1}^N d\mu(y_{r+1}) \leq \int V_{r+1}^N d\mu(y_{r+1}),$$

so

$$\lim_{N \rightarrow \infty} R_r^N \leq \lim_{N \rightarrow \infty} \int V_{r+1}^N d\mu(y_{r+1}) = \int V_{r+1} d\mu(y_{r+1})$$

by the Lebesgue theorem on monotone convergence. Thus,

$$\lim_{N \rightarrow \infty} R_r^N \leq \min_{x_{r+1}} \int V_{r+1} d\mu(y_{r+1}) = R_r. \tag{59}$$

On the other hand, for any  $N \geq 1$ ,

$$\int V_{r+1}^N d\mu(y_{r+1}) \geq \int V_{r+1} d\mu(y_{r+1}),$$

so

$$R_r^N = \min_{x_{r+1}} \int V_{r+1}^N d\mu(y_{r+1}) \geq \min_{x_{r+1}} \int V_{r+1} d\mu(y_{r+1}) = R_r,$$

hence

$$\lim_{N \rightarrow \infty} R_r^N \geq R_r.$$

From this and (59), we get that

$$\lim_{N \rightarrow \infty} R_r^N = R_r.$$

Therefore, from (57) it follows that

$$V_r = \lim_{N \rightarrow \infty} V_r^N = \min\{l_r, f_{\theta_0}^r + R_r\},$$

which proves (54). □

Let us note now that the right-hand side of (56) coincides with

$$\inf_{(\chi, \psi) \in F} L(\chi, \psi).$$

**Lemma 5.4.**

$$\inf_{(\chi, \psi) \in F} L(\chi, \psi) = 1 + R_0. \tag{60}$$

*Proof.* Let us denote

$$U = \inf_{(\chi, \psi) \in \mathbb{F}} L(\chi, \psi), \quad U_N = 1 + R_0^N.$$

By Theorem 3, for any  $N = 1, 2, \dots$

$$U_N = \inf_{(\chi, \psi): \psi \in \Delta^N} L(\chi, \psi).$$

Obviously,  $U_N \geq U$  for any  $N = 1, 2, \dots$ , so

$$\lim_{N \rightarrow \infty} U_N \geq U. \tag{61}$$

Let us show first that in fact there is an equality in (61).

Suppose the contrary, i.e. that  $\lim_{N \rightarrow \infty} U_N = U + 4\epsilon$ , with some  $\epsilon > 0$ . We immediately have from this that

$$U_N \geq U + 3\epsilon \tag{62}$$

for all sufficiently large  $N$ .

On the other hand, by the definition of  $U$  there exists a  $\psi$  such that  $U \leq L(\chi, \psi) \leq U + \epsilon$  and  $(\chi, \psi) \in \mathbb{F}$ .

Because, by Lemma 5.1,  $L_N(\chi, \psi) \rightarrow L(\chi, \psi)$ , as  $N \rightarrow \infty$ , we have that

$$L_N(\chi, \psi) \leq U + 2\epsilon \tag{63}$$

for all sufficiently large  $N$  as well. Because, by definition,  $L_N(\chi, \psi) \geq U_N$ , we have that

$$U_N \leq U + 2\epsilon$$

for all sufficiently large  $N$ , which contradicts (62).

Thus,

$$\lim_{N \rightarrow \infty} U_N = U.$$

Now, to get (60) we note first that

$$\begin{aligned} U &= \lim_{N \rightarrow \infty} U_N = 1 + \lim_{N \rightarrow \infty} \inf_{x_1} \int V_1^N(x_1; y_1) d\mu(y_1) \\ &\leq 1 + \inf_{x_1} \int V_1(x_1; y_1) d\mu(y_1) = 1 + R_0. \end{aligned}$$

On the other hand, by Theorem 5.3,

$$U = \inf_{(\chi, \psi) \in \mathbb{F}} L(\chi, \psi) \geq 1 + R_0,$$

thus,

$$U = 1 + R_0.$$

□

The following theorem characterizes the structure of the control- and the stopping-part of optimal sequential testing procedures.

**Theorem 5.5.** *If there is a strategy  $(\chi, \psi) \in F$  such that*

$$L(\chi, \psi) = \inf_{(\chi', \psi') \in F} L(\chi', \psi'), \tag{64}$$

then

$$I_{\{\chi_k^X < f_{\theta_0}^{k,X} + R_k^X\}} \leq \psi_k^X \leq I_{\{\chi_k^X \leq f_{\theta_0}^{k,X} + R_k^X\}} \tag{65}$$

$\mu^k$ -almost everywhere on  $C_k^{\psi, \chi}$ , and

$$\int V_{k+1}^X d\mu(y_{k+1}) = R_k^X \tag{66}$$

$\mu^k$ -almost everywhere on  $\bar{C}_k^{\psi, \chi}$ , for any  $k = 1, 2, \dots$ , where  $\chi_1$  is defined in such a way that

$$\int V_1^X d\mu(y_1) = R_0. \tag{67}$$

On the other hand, if a strategy  $(\psi, \chi)$  satisfies (65)  $\mu^k$ -almost everywhere on  $C_k^{\psi, \chi}$ , and satisfies (66)  $\mu^k$ -almost everywhere on  $\bar{C}_k^{\psi, \chi}$ , for any  $k = 1, 2, \dots$ , where  $\chi_1$  is such that (67) is fulfilled, then  $(\psi, \chi) \in F$ , and (64) holds.

*Proof.* Let  $(\chi, \psi) \in F$  be any strategy. By Theorem 5.3 for any fixed  $r \geq 1$  the following inequalities hold:

$$L(\chi, \psi) \geq \sum_{n=1}^r \int s_n^{\psi, \chi} (n f_{\theta_0}^{n, \chi} + l_n^X) d\mu^n + \int c_n^{\psi, \chi} ((r+1) f_{\theta_0}^{r+1, \chi} + V_{r+1}^X) d\mu^{r+1} \tag{68}$$

$$\geq \sum_{n=1}^{r-1} \int s_n^{\psi, \chi} (n f_{\theta_0}^{n, \chi} + l_n^X) d\mu^n + \int c_r^{\psi, \chi} (r f_{\theta_0}^{r, \chi} + V_r^X) d\mu^r \tag{69}$$

$$\geq \dots \tag{70}$$

$$\geq \int \psi_1^X (f_{\theta_0}^{1, \chi} + l_1^X) d\mu^1 + \int (1 - \psi_1^X) (2 f_{\theta_0}^{2, \chi} + V_2^X) d\mu^2$$

$$\geq 1 + \int V_1^X d\mu \geq 1 + R_0. \tag{71}$$

Let us suppose that the right-hand side of (71) is attained by some  $(\chi, \psi) \in F$ . This means that there are equalities in *all* of the inequalities (68) - (71). Then, first of all, we get that

$$R_0 = \int V_1^X(y_1) d\mu(y_1), \tag{72}$$

and, successively for  $k = 1, 2, \dots$ , each time applying Lemma 4.1, that

$$I_{\{l_k^x < f_{\theta_0}^{k,x} + R_k^x\}} \leq \psi_k^x \leq I_{\{l_k^x \leq f_{\theta_0}^{k,x} + R_k^x\}} \tag{73}$$

$\mu^k$ -almost everywhere on  $C_k^{\psi, x}$ , and

$$\int V_{k+1}^x(y^{(k+1)}) d\mu(y_{k+1}) = R_k^x, \tag{74}$$

$\mu^k$ -almost everywhere on  $\bar{C}_k^{\psi, x}$ . The first part of Theorem 5.5 is proved.

To prove the second part, let us suppose that  $(\chi, \psi)$  satisfies (72) - (74). Applying Lemma 4.1, we see that all the inequalities in (69)-(71) are in fact equalities for  $\psi = \psi^r = (\psi_1, \dots, \psi_r, 1, \dots)$ .

In particular, this means that there exists

$$\lim_{r \rightarrow \infty} \left[ \sum_{n=1}^r \int s_n^{\psi, x} (n f_{\theta_0}^{n,x} + l_n^x) d\mu^n + \int c_{r+1}^{\psi, x} ((r+1) f_{\theta_0}^{r+1,x} + V_{r+1}^x) d\mu^{r+1} \right] = 1 + R_0. \tag{75}$$

From this, it follows immediately that there exists as well

$$\lim_{r \rightarrow \infty} \sum_{n=1}^r \int s_n^{\psi, x} (n f_{\theta_0}^{n,x} + l_n^x) d\mu^n \leq 1 + R_0, \tag{76}$$

and that

$$\limsup_{r \rightarrow \infty} \int c_r^{\psi, x} r f_{\theta_0}^{r,x} d\mu^r = \limsup_{r \rightarrow \infty} (r P_{\theta_0}^x(\tau_\psi \geq r)) < \infty. \tag{77}$$

From (77) it follows that  $P_{\theta_0}^x(\tau_\psi \geq r) \rightarrow 0$ , as  $r \rightarrow \infty$ , i.e. that  $(\chi, \psi) \in F$ . Now, the left-hand side of (76) is  $L(\chi, \psi)$  (because  $(\chi, \psi) \in F$ ), and hence

$$L(\chi, \psi) \leq 1 + R_0. \tag{78}$$

On the other hand, by virtue of (68) - (71)  $L(\chi, \psi) \geq 1 + R_0$ . From this, and (78), we see that  $L(\chi, \psi) = 1 + R_0$ . Because, by Lemma 5.4,

$$\inf_{(\chi', \psi') \in F} L(\chi', \psi') = 1 + R_0,$$

this proves the second part of Theorem 5.5. □

**Remark 5.6.** *Theorem 5.5 treats the optimality among strategies which take at least one observation. If we allow not to take any observation, there is a possibility that the trivial testing procedure (see Remark 4.5) gives a better result. It is easy to see that this happens if*

$$\min\{\lambda_0, \lambda_1\} < 1 + R_0.$$

**Remark 5.7.** *In a particular case when the control variable takes only one value,  $x$ , Theorem 5.5 characterizes the optimal stopping rule in the problem of testing two simple hypotheses for independent identically distributed (with density  $f_\theta(y|x)$ ) observations (see [2], [3], [4]). It is very well known that the optimal stopping rule is based, in this particular case, on the likelihood ratio statistic (and the resulting test is known as the Sequential Probability Ratio Test (SPRT) [5]). Because of this, we will dedicate the following section to finding a likelihood structure of the optimal stopping rule in Theorem 5.5, in the general case of non-trivial control variables.*

## 6 Likelihood Ratio Structure of Optimal Strategy

In this section, we will give to the optimal strategy in Theorem 5.5 an equivalent form related to the likelihood ratio process.

Let us start with defining the likelihood ratio:

$$Z_n = Z_n(x^{(n)}, y^{(n)}) = \prod_{i=1}^n \frac{f_{\theta_1}(y_i|x_i)}{f_{\theta_0}(y_i|x_i)}.$$

Let us introduce then the following sequence of functions:

$$\rho_0(z) = g(z) \equiv \min\{\lambda_0, \lambda_1 z\}, \quad (79)$$

and for  $k = 1, 2, 3, \dots$ :

$$\rho_k(z) = \min \left\{ g(z), 1 + \min_x \int f_{\theta_0}(y|x) \rho_{k-1} \left( z \frac{f_{\theta_1}(y|x)}{f_{\theta_0}(y|x)} \right) d\mu(y) \right\} \quad (80)$$

(we are supposing that all  $\rho_k$ ,  $k = 0, 1, 2, \dots$  are well-defined and measurable functions of  $z$ ). It is easy to see that (see (39), (40))

$$V_N^N = f_{\theta_0}^N \rho_0(Z_N),$$

and for  $k = N - 1, N - 2, \dots, 1$

$$V_k^N = f_{\theta_0}^k \rho_{N-k}(Z_k). \quad (81)$$

It is not difficult to see (very much like in Lemma 5.2) that

$$\rho_k(z) \geq \rho_{k+1}(z)$$

for any  $k = 0, 1, 2, \dots$ , so there exists

$$\rho(z) = \lim_{n \rightarrow \infty} \rho_n(z). \quad (82)$$

Using arguments similar to those used in the proof of Theorem 5.3, it can be shown, starting from (80), that

$$\rho(z) = \min \{g(z), 1 + R(z)\}, \tag{83}$$

where

$$R(z) = \min_x \int f_{\theta_0}(y|x) \rho \left( z \frac{f_{\theta_1}(y|x)}{f_{\theta_0}(y|x)} \right) d\mu(y). \tag{84}$$

Let us pass now to the limit, as  $N \rightarrow \infty$ , in (81). We see that

$$V_k = f_{\theta_0}^k \rho(Z_k).$$

Using these expressions in Theorem 5.5 we get

**Theorem 6.1.** *If there exists a strategy  $(\chi, \psi) \in F$  such that*

$$L(\chi, \psi) = \inf_{(\chi', \psi') \in F} L(\chi', \psi'), \tag{85}$$

then

$$I_{\{g(Z_k^X) < 1 + R(Z_k^X)\}} \leq \psi_k^X \leq I_{\{g(Z_k^X) \leq 1 + R(Z_k^X)\}} \tag{86}$$

$P_{\theta_0}^X$ -almost sure on

$$\{y^{(k)} : (1 - \psi_1^X(y^{(1)})) \dots (1 - \psi_{k-1}^X(y^{(k-1)})) > 0\}, \tag{87}$$

and

$$\int f_{\theta_0}(y|\chi_{k+1}) \rho \left( Z_k^X \frac{f_{\theta_1}(y|\chi_{k+1})}{f_{\theta_0}(y|\chi_{k+1})} \right) d\mu(y) = R(Z_k^X) \tag{88}$$

$P_{\theta_0}^X$ -almost sure on

$$\{y^{(k)} : (1 - \psi_1^X(y^{(1)})) \dots (1 - \psi_k^X(y^{(k)})) > 0\}, \tag{89}$$

where  $\chi_1$  is defined in such a way that

$$\int f_{\theta_0}(y|\chi_1) \rho \left( \frac{f_{\theta_1}(y|\chi_1)}{f_{\theta_0}(y|\chi_1)} \right) d\mu(y) = R(1). \tag{90}$$

On the other hand, if  $(\chi, \psi)$  satisfies (86)  $P_{\theta_0}^X$ -almost sure on (87) and satisfies (88)  $P_{\theta_0}^X$ -almost sure on (89), for any  $k = 1, 2, \dots$ , where  $\chi_1$  satisfies (90), then  $(\chi, \psi) \in F$  and  $(\chi, \psi)$  satisfies (85).

**Remark 6.2.** *It is not difficult to see (very much like in [4]) that when*

$$1 + R(\infty) = 1 + \lim_{z \rightarrow \infty} R(z) > \lambda_0, \quad (91)$$

*there exist  $0 < A < B < \infty$  such that  $g(z) > 1 + R(z)$  (see (86)) is equivalent to  $z \in (A, B)$ . By Theorem 7.1, this implies, in particular, that the optimal stopping rule is of an SPRT type: stopping occurs when  $Z_n^\chi$  for the first time exits an interval. Nevertheless, unlike the classical problem of sequential testing, this does not help very much in this case of a statistical experiment with control, because an essential part of the problem is the construction of the optimal control rule (see (88)), and there is no apparent way to relate it to the stopping constants  $A$  and  $B$ .*

*If (91) does not hold, the optimal stopping rule is still simple, but may seem somewhat strange. For example, if  $1 + R(\infty) < \lambda_0$ , then the optimal strategy prescribes to stop when, for the first time,  $Z_n^\chi$  drops below some  $A > 0$ , and accept  $H_0$  at that time. In this case, obviously, the experiment may continue indefinitely, with a large probability, if the alternative hypothesis is true. This does not make much practical sense, and we are not sure that this may ever happen in any testing problem with non-trivial control, but we are unable, generally speaking, to prove that (91) is always fulfilled.*

*The reason why the optimal stopping time may not have a finite expectation under one of the hypotheses lies in the definition of the error probabilities (2) and (3) that do not penalize continuing the experiment indefinitely, and/or in the fact that the average sample number under the alternative hypothesis is not taken into account when minimizing the "risk" (see definition of  $L(\chi, \psi)$  in (7)). Similar phenomena occur even in the "no-control" case and even when the observations are independent and identically distributed, if the average sample number under one of the hypotheses is disregarded as a criterion of optimization (see [1]). Taking into account the average sample number under both the null- and the alternative hypothesis remedies this problem (see Remark 6.3) below.*

**Remark 6.3.** *Considering as a criterion of optimization, instead of  $N(\chi, \psi) = E_{\theta_0}^\chi \tau_\psi$  in (7), a weighted sum of the two average sample numbers:*

$$N(\chi, \psi) = \pi_0 E_{\theta_0}^\chi \tau_\psi + \pi_1 E_{\theta_1}^\chi \tau_\psi,$$

*where  $\pi_0$  and  $\pi_1$  are some positive numbers, leads to a Bayesian problem of sequential testing in the present context. There are almost evident modifications of Theorems 4.2, 5.3, 5.5 and 6.1 giving solutions to the respective Bayesian problems as well. For example, instead of (39) it should be used*

$$V_{k-1}^N = \min\{l_{k-1}, \pi_0 f_{\theta_0}^{k-1} + \pi_1 f_{\theta_1}^{k-1} + R_{k-1}^N\}, \quad (92)$$

(41) should be modified to

$$I_{\{t_k^x < \pi_0 f_{\theta_0}^{k,x} + \pi_1 f_{\theta_1}^{k,x} + R_k^{N,x}\}} \leq \psi_k^x \leq I_{\{t_k^x \leq \pi_0 f_{\theta_0}^{k,x} + \pi_1 f_{\theta_1}^{k,x} + R_k^{N,x}\}}, \tag{93}$$

etc., etc.

## 7 Application to the Conditional Problem

In this section, we apply the results obtained in the preceding sections to minimizing the average sample size  $N(\chi, \psi) = E_{\theta_0}^x \tau_\psi$  over all sequential testing procedures with error probabilities not exceeding some prescribed levels.

Combining Theorems 2.1, 3.1 and 6.1, we immediately have the following

**Theorem 7.1.** *Let  $(\chi, \psi)$  satisfy the conditions of Theorem 6.1, and let  $\phi$  be defined by*

$$\phi_n = I_{\{\lambda_0 f_{\theta_0}^n \leq \lambda_1 f_{\theta_1}^n\}} \tag{94}$$

for  $n = 1, 2, \dots$

Then for any sequential testing procedure  $(\chi', \psi', \phi')$  such that

$$\alpha(\chi', \psi', \phi') \leq \alpha(\chi, \psi, \phi) \quad \text{and} \quad \beta(\chi', \psi', \phi') \leq \beta(\chi, \psi, \phi) \tag{95}$$

it holds

$$N(\chi', \psi') \geq N(\chi, \psi). \tag{96}$$

The inequality in (96) is strict if at least one of the inequalities in (95) is strict.

If there are equalities in all of the inequalities in (95) and (96), then  $(\chi', \psi')$  satisfies the conditions of Theorem 6.1 as well (with  $\chi'$  instead of  $\chi$  and  $\psi'$  instead of  $\psi$ ).

*Proof.* The only thing to be proved is the last assertion.

Let us suppose that

$$\alpha(\chi', \psi', \phi') = \alpha(\chi, \psi, \phi),$$

$$\beta(\chi', \psi', \phi') = \beta(\chi, \psi, \phi),$$

and

$$N(\chi', \psi') = N(\chi, \psi).$$

Then, obviously,

$$L(\chi, \psi, \phi) = L(\chi, \psi) = L(\chi', \psi', \phi') \geq L(\chi', \psi') \tag{97}$$

(see (7)) and Remark 3.3.

By Theorem 6.1, there can not be a strict inequality in the last inequality in (97), so  $L(\chi, \psi) = L(\chi', \psi')$ . From Theorem 6.1 it follows now that  $(\chi', \psi')$  satisfies (86) – (90) as well.  $\square$

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