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On Goodman-Rønning-Type Harmonic Univalent Functions Defined by Ruscheweyh Operator

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Abstract

In this paper, we use Ruscheweyh derivative operator to introduce and study a class of Goodman-Rønning-type harmonic univalent functions. We obtain coefficient conditions, extreme points, distortion bounds, convolution conditions, convex combination for the aforementioned class. Further, result on integral operators is also given.

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1 Introduction

A continuous function $f = u+iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $\mathcal{D} \subset \mathbb{C}$, we can write $f(z) = h + \overline{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$. See Clunie and Sheil-Small [1].

Denote by $\mathcal{H}$ the family of functions $f = h+\overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. For $f = h + \overline{g} \in \mathcal{H}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k \qquad |b_1| < 1. \tag{1.1}$$

Let $\mathcal{T}$ be defined as the subclass of $\mathcal{H}$ consisting of all functions $f = h + \overline{g}$ where h and g are given by

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \qquad g(z) = \sum_{k=1}^{\infty} |b_k| z^k \qquad |b_1| < 1. \tag{1.2}$$

In 1984, Clunie and Sheil-Small [1] investigated the class $\mathcal{H}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $\mathcal{H}$ and its subclasses which include the ones studied by Silverman [5], Silverman and Silvia [6] and, Jahangiri [7].

Definition 1.1 *[2] The subclass $G_H(\alpha) \subset \mathcal{H}$ consisting of harmonic univalent functions f satisfying the following condition*

$$Re\left\{(1 + e^{i\gamma})\frac{zf'(z)}{f(z)} - e^{i\gamma}\right\} \geq \alpha, \qquad 0 \leq \alpha < 1,\ \gamma \in \mathbb{R}.$$

In [2], they proved that if $f = h + \overline{g}$ is given by (1.1) and if

$$\sum_{k=1}^{\infty} \left[\frac{2k - 1 - \alpha}{1 - \alpha}|a_k| + \frac{2k + 1 + \alpha}{1 - \alpha}|b_k|\right] \leq 2, \qquad a_1 = 1,\ 0 \leq \alpha < 1, \tag{1.3}$$

then f is harmonic univalent functions in $\mathbb{U}$ and $f \in G_H(\alpha)$. This condition is proved to be necessary if $f = h + \overline{g}$ is given by (1.2).

Now, we define the Ruscheweyh derivative operator D^n:

Definition 1.2 *[3] Let $\phi(z) = z + \sum_{k=2}^{\infty} \phi_k z^k$. Then for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$*

$$D^n\phi(z) = \frac{z}{(1 - z)^{n+1}} * \phi(z) = z + \sum_{k=2}^{\infty} C(n, k)\phi_k z^k$$

where $C(n, k) = \binom{k+n-1}{n}$ and $*$ denotes the Hadamard product of two analytic functions.

Murugusundaramoorthy [4] has introduced the modified Ruscheweyh derivatives operator of harmonic univalent function $f = h + \overline{g} \in \mathcal{H}$ as

$$D^n f(z) = D^n h(z) + \overline{D^n g(z)}, \qquad n \in \mathbb{N}_0, \tag{1.4}$$

where

$$D^n h(z) = z + \sum_{k=2}^{\infty} C(n,k) a_k z^k \quad \text{and} \quad D^n g(z) = \sum_{k=2}^{\infty} C(n,k) b_k z^k.$$

Definition 1.3 *Let $M_{\mathcal{H}}(n,\alpha)$ denote the family of harmonic functions f of the form (1.1) such that*

$$Re\left\{(1+e^{i\gamma})\frac{D^{n+1}f(z)}{D^n f(z)} - e^{i\gamma}\right\} \geq \alpha, \tag{1.5}$$

where $0 \leq \alpha < 1$, $\gamma \in \mathbb{R}$ and $D^n f(z)$ is defined by (1.4).

Note that for the case $n = 0$, was given in [2]. Also note that for the case $n = 0$, $\alpha = 0$, was given by Silverman and Silvia [5] and for $n = 0$, $\alpha = b_1 = 0$, see [6]. Further note that if $n = 0$ and the co-analytic part of $f = h + \overline{g}$ is zero, then the class $M_{\mathcal{H}}(n,\alpha)$ reduces to the class studied in [8] (see also [9]).
We further denote by $T_{\mathcal{H}}(n,\alpha)$ the subclass of $M_{\mathcal{H}}(n,\alpha)$, where $T_{\mathcal{H}}(n,\alpha) = \mathcal{T} \cap M_{\mathcal{H}}(n,\alpha)$.

2 Coefficient Bounds

In this section, we introduce a sufficient coefficient bound for harmonic functions in $M_{\mathcal{H}}(n,\alpha)$ and $T_{\mathcal{H}}(n,\alpha)$.

Theorem 2.1 *Let $f = h + \overline{g}$ be given by (1.1). If*

$$\sum_{k=1}^{\infty} [(2k-\alpha-1)|a_k| + (2k+\alpha+1)|b_k|] C(n,k) \leq 2(1-\alpha), \tag{2.1}$$

where $a_1 = 1$, $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$, then $f(z)$ is sense-preserving harmonic univalent in $\mathbb{U}$, and $f \in M_{\mathcal{H}}(n,\alpha)$.

Proof. Let $|z_1| \le |z_2| < 1$,

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\ge 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\ &= 1-\left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k-z_2^k)}{(z_1-z_2)+\sum_{k=2}^{\infty} a_k(z_1^k-z_2^k)}\right| \\ &> 1-\frac{\sum_{k=1}^{\infty} k|b_k|}{1-\sum_{k=2}^{\infty} k|a_k|} \\ &\ge 1-\frac{\sum_{k=1}^{\infty} \frac{(2k+\alpha+1)C(n,k)}{1-\alpha}|b_k|}{1-\sum_{k=2}^{\infty} \frac{(2k-\alpha-1)C(n,k)}{1-\alpha}|a_k|} \ge 0.\end{aligned}$$

Consequently, f is univalent in $\mathbb{U}$. We note that f is sense preserving in $\mathbb{U}$. This is because

$$\begin{aligned}|h'(z)| &\ge 1-\sum_{k=2}^{\infty} k|a_k||z|^{k-1} > 1-\sum_{k=2}^{\infty} k|a_k| \ge 1-\sum_{k=2}^{\infty}\frac{(2k-\alpha-1)C(n,k)}{1-\alpha}|a_k||z| \\ &\ge \sum_{k=1}^{\infty}\frac{(2k+\alpha+1)C(n,k)}{1-\alpha}|b_k| > \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \ge |g'(z)|.\end{aligned}$$

Now according to the condition (1.5) we only need to show that if (2.1) holds then

$$\begin{aligned}\operatorname{Re}\left\{(1+e^{i\gamma})\frac{D^{n+1}f(z)}{D^n f(z)}-e^{i\gamma}\right\} &= \operatorname{Re}\left\{(1+e^{i\gamma})\frac{D^{n+1}h(z)-\overline{D^{n+1}g(z)}}{D^n h(z)+\overline{D^n g(z)}}-e^{i\gamma}\right\} \\ &\ge \alpha.\end{aligned}$$

Using the fact that $\operatorname{Re} w \ge \alpha$ if and only if $|1+w-\alpha| \ge |1-w+\alpha|$, it suffices to show that

$$\begin{aligned}&\left|(1-\alpha)D^n f(z)+(1+e^{i\gamma})D^{n+1}f(z)-e^{i\gamma}D^n f(z)\right| \\ &\qquad -\left|(1+\alpha)D^n f(z)-(1+e^{i\gamma})D^{n+1}f(z)+e^{i\gamma}D^n f(z)\right| \ge 0. \end{aligned} \tag{2.2}$$

Substituting for $D^n f(z)$ and $D^{n+1}f(z)$ in (2.2) yields

$$
\begin{aligned}
& \left|(1-\alpha)D^n f(z)+(1+e^{i\gamma})D^{n+1}f(z)-e^{i\gamma}D^n f(z)\right| \\
& \qquad -\left|(1+\alpha)D^n f(z)-(1+e^{i\gamma})D^{n+1}f(z)+e^{i\gamma}D^n f(z)\right| \\
= & \left|(2-\alpha)z+\sum_{k=2}^{\infty}(k+ke^{i\gamma}+1-\alpha-e^{i\gamma})C(n,k)a_k z^k \right. \\
& \left. \qquad -\sum_{k=1}^{\infty}(k+ke^{i\gamma}-1+\alpha+e^{i\gamma})C(n,k)\overline{b_k z^k}\right|
\end{aligned}
$$

$$
\begin{aligned}
& -\left|\alpha z-\sum_{k=2}^{\infty}(k+ke^{i\gamma}-1-\alpha-e^{i\gamma})C(n,k)a_k z^k \right. \\
& \left. \qquad +\sum_{k=1}^{\infty}(k+ke^{i\gamma}+1+\alpha+e^{i\gamma})C(n,k)\overline{b_k z^k}\right| \\
\geq\ & (2-\alpha)|z|-\sum_{k=2}^{\infty}(2k-\alpha)C(n,k)|a_k||z|^k-\sum_{k=1}^{\infty}(2k+\alpha)C(n,k)|b_k||z|^k \\
& \qquad -\alpha|z|-\sum_{k=2}^{\infty}(2k-\alpha-2)C(n,k)|a_k||z|^k-\sum_{k=1}^{\infty}(2k+\alpha+2)C(n,k)|b_k||z|^k \\
=\ & 2(1-\alpha)|z|-2\sum_{k=2}^{\infty}(2k-\alpha-1)C(n,k)|a_k||z|^k-2\sum_{k=1}^{\infty}(2k+\alpha+1)C(n,k)|b_k||z|^k \\
=\ & 2(1-\alpha)|z|\left\{1-\sum_{k=2}^{\infty}\frac{(2k-\alpha-1)C(n,k)}{1-\alpha}|a_k||z|^{k-1}-\sum_{k=1}^{\infty}\frac{(2k+\alpha+1)C(n,k)}{1-\alpha}|b_k||z|^{k-1}\right\} \\
>\ & 2(1-\alpha)\left\{1-\left(\sum_{k=2}^{\infty}\frac{(2k-\alpha-1)C(n,k)}{1-\alpha}|a_k|+\sum_{k=1}^{\infty}\frac{(2k+\alpha+1)C(n,k)}{1-\alpha}|b_k|\right)\right\}.
\end{aligned}
$$

This last expression is non-negative by (2.1), and so the proof is complete.

The harmonic function

$$
f(z)=z+\sum_{k=2}^{\infty}\frac{1-\alpha}{(2k-\alpha-1)C(n,k)}x_k z^k+\sum_{k=1}^{\infty}\frac{1-\alpha}{(2k+\alpha+1)C(n,k)}\overline{y_k z^k} \tag{2.3}
$$

where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.3) are in $M_{\mathcal{H}}(n,\alpha)$ because

$$
\sum_{k=1}^{\infty}\left(\frac{(2k-\alpha-1)C(n,k)}{1-\alpha}|a_k|+\frac{(2k+\alpha+1)C(n,k)}{1-\alpha}|b_k|\right)=1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=2.
$$

The restriction placed in Theorem 2.1 on the moduli of the coefficients of $f = h + \overline{g}$ enables us to conclude for arbitrary rotation of the coefficients of f that the resulting functions would still be harmonic univalent and $f \in M_{\mathcal{H}}(n, \alpha)$. We next show that the condition (2.1) is also necessary for functions in $T_{\mathcal{H}}(n, \alpha)$.

Theorem 2.2 *Let $f = h + \overline{g}$ be given by (1.2). Then $f \in T_{\mathcal{H}}(n, \alpha)$ if and only if*

$$\sum_{k=1}^{\infty} [(2k - \alpha - 1)|a_k| + (2k + \alpha + 1)|b_k|]C(n, k) \leq 2(1 - \alpha), \tag{2.4}$$

where $a_1 = 1$, $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$.

Proof. The *if* part follows from Theorem 2.1 upon noting that if h and g in $f = h + \overline{g} \in M_{\mathcal{H}}(n, \alpha)$ are of the form (1.2) then $f \in T_{\mathcal{H}}(n, \alpha)$. For the *onlyif* part, we show that $f \notin T_{\mathcal{H}}(n, \alpha)$ if the condition (2.4) does not hold. Note that a necessary and sufficient condition for $f = h + \overline{g}$ given by (1.2) to be in $T_{\mathcal{H}}(n, \alpha)$ is that

$$\begin{aligned}
&\operatorname{Re}\left\{(1 + e^{i\gamma})\frac{D^{n+1} f(z)}{D^n f(z)} - e^{i\gamma}\right\} \\
&\qquad = \operatorname{Re}\left\{(1 + e^{i\gamma})\frac{D^{n+1} h(z) - \overline{D^{n+1} g(z)}}{D^n h(z) + \overline{D^n g(z)}} - e^{i\gamma}\right\} \geq \alpha.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\operatorname{Re}\left\{\frac{(1 - \alpha)z - \sum_{k=2}^{\infty} [k - \alpha + (n - 1)e^{i\gamma}]C(n, k)|a_k|z^k}{z - \sum_{k=2}^{\infty} C(n, k)|a_k|z^k + \sum_{k=1}^{\infty} C(n, k)|b_k|\overline{z}^k}\right. \\
&\qquad\qquad \left. - \frac{\sum_{k=1}^{\infty} [k + \alpha + (k + 1)e^{i\gamma}]C(n, k)|b_k|\overline{z}^k}{z - \sum_{k=2}^{\infty} C(n, k)|a_k|z^k + \sum_{k=1}^{\infty} C(n, k)|b_k|\overline{z}^k}\right\} \\
= \; &\operatorname{Re}\left\{\frac{1 - \alpha - \sum_{k=2}^{\infty} [k - \alpha + (k - 1)e^{i\gamma}]C(n, k)|a_k|z^{k-1}}{1 - \sum_{k=2}^{\infty} C(n, k)|a_k|z^{k-1} + \frac{\overline{z}}{z}\sum_{k=1}^{\infty} C(n, k)|b_k|\overline{z}^{k-1}}\right. \\
&\qquad\qquad \left. - \frac{\frac{\overline{z}}{z}\sum_{k=1}^{\infty} [k + \alpha + (k + 1)e^{i\gamma}]C(n, k)|b_k|\overline{z}^{k-1}}{1 - \sum_{k=2}^{\infty} C(n, k)|a_k|z^{k-1} + \frac{\overline{z}}{z}\sum_{k=1}^{\infty} C(n, k)|b_k|\overline{z}^{k-1}}\right\} \\
\geq \; & 0.
\end{aligned}$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis, where $0 \leq z = r < 1$, we must have

$$\mathrm{Re}\left\{\frac{1-\alpha-\sum_{k=2}^{\infty}(k-\alpha)C(n,k)|a_k|r^{k-1}-\sum_{k=1}^{\infty}(k+\alpha)C(n,k)|b_k|r^{k-1}}{1-\sum_{k=2}^{\infty}C(n,k)|a_k|r^{k-1}+\sum_{k=1}^{\infty}C(n,k)|b_k|r^{k-1}}\right.$$
$$\left.-e^{i\gamma}\frac{\sum_{k=2}^{\infty}(k-1)C(n,k)|a_k|r^{k-1}+\sum_{k=1}^{\infty}(k+1)C(n,k)|b_k|r^{k-1}}{1-\sum_{k=2}^{\infty}C(n,k)|a_k|r^{k-1}+\sum_{k=1}^{\infty}C(n,k)|b_k|r^{k-1}}\right\} \geq 0.$$

Since $\mathrm{Re}(-e^{i\gamma}) \geq -1$, the above inequality reduces to

$$\frac{1-\alpha-\sum_{k=2}^{\infty}(2k-\alpha-1)C(n,k)|a_k|r^{k-1}-\sum_{k=1}^{\infty}(2k+\alpha+1)C(n,k)|b_k|r^{k-1}}{1-\sum_{k=2}^{\infty}C(n,k)|a_k|r^{k-1}+\sum_{k=1}^{\infty}C(n,k)|b_k|r^{k-1}} \geq 0. \tag{2.5}$$

If the condition (2.4) does not hold then the numerator in (2.5) is negative for r sufficiently close to 1. Thus there exists a $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.5) is negative. This contradicts the condition for $f \in T_{\mathcal{H}}(n, \alpha)$ and hence the result.

3 Distortion Bounds and Extreme Points

In this section, we shall obtain distortion bounds for functions in $T_{\mathcal{H}}(n, \alpha)$ and also provide extreme points for this class.

Theorem 3.1 *If $f \in T_{\mathcal{H}}(n, \alpha)$, for $0 \leq \alpha < 0$, $n \in \mathbb{N}_0$ and $|z| = r < 1$, then*

$$|f(z)| \leq (1+|b_1|)r + \frac{1}{C(n,2)}\left(\frac{1-\alpha}{3-\alpha} - \frac{3+\alpha}{3-\alpha}|b_1|\right)r^2$$

and

$$|f(z)| \geq (1-|b_1|)r - \frac{1}{C(n,2)}\left(\frac{1-\alpha}{3-\alpha} - \frac{3+\alpha}{3-\alpha}|b_1|\right)r^2.$$

Proof. Let $f \in T_{\mathcal{H}}(n,\alpha)$. Taking the absolute value of f we obtain

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)|z| + \sum_{k=2}^{\infty}|a_k||z|^k + \sum_{k=2}^{\infty}|b_k||\overline{z}|^k \\
&\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\
&\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2 \\
&\leq (1+|b_1|)r + \frac{1-\alpha}{(3-\alpha)C(n,2)}\sum_{k=2}^{\infty}\frac{(3-\alpha)C(n,2)}{1-\alpha}(|a_k|+|b_k|)r^2 \\
&\leq (1+|b_1|)r \\
&\quad + \frac{1-\alpha}{(3-\alpha)C(n,2)}\sum_{k=2}^{\infty}\left(\frac{(2k-\alpha-1)C(n,k)}{1-\alpha}|a_k| + \frac{(2k+\alpha+1)C(n,k)}{1-\alpha}|b_k|\right)r^2 \\
&\leq (1+|b_1|)r + \frac{1-\alpha}{(3-\alpha)C(n,2)}\left(1-\frac{3+\alpha}{1-\alpha}|b_1|\right)r^2 \\
&\leq (1+|b_1|)r + \frac{1}{C(n,2)}\left(\frac{1-\alpha}{3-\alpha}-\frac{3+\alpha}{3-\alpha}|b_1|\right)r^2.
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &\geq (1-|b_1|)|z| - \sum_{k=2}^{\infty}|a_k||z|^k + \sum_{k=2}^{\infty}|b_k||\overline{z}|^k \\
&\geq (1-|b_1|)r - \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\
&\geq (1-|b_1|)r - \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2 \\
&\geq (1-|b_1|)r - \frac{1-\alpha}{(3-\alpha)C(n,2)}\sum_{k=2}^{\infty}\frac{(3-\alpha)C(n,2)}{1-\alpha}(|a_k|+|b_k|)r^2 \\
&\geq (1-|b_1|)r \\
&\quad - \frac{1-\alpha}{(3-\alpha)C(n,2)}\sum_{k=2}^{\infty}\left(\frac{(2k-\alpha-1)C(n,k)}{1-\alpha}|a_k| + \frac{(2k+\alpha+1)C(n,k)}{1-\alpha}|b_k|\right)r^2 \\
&\geq (1-|b_1|)r - \frac{1-\alpha}{(3-\alpha)C(n,2)}\left(1-\frac{3+\alpha}{1-\alpha}|b_1|\right)r^2 \\
&\geq (1-|b_1|)r - \frac{1}{C(n,2)}\left(\frac{1-\alpha}{3-\alpha}-\frac{3+\alpha}{3-\alpha}|b_1|\right)r^2.
\end{aligned}$$

The bounds given in Theorem 3.1 for the functions $f = h + \overline{g}$ where h and g of the form (1.2) also hold for functions of the form (1.1) if the coefficient condition (2.1) is satisfied.

The following covering result follows from the second inequality in Theorem 3.1.

Corollary 3.2 *If $f \in T_{\mathcal{H}}(n, \alpha)$, then*

$$\left\{ w : |w| < \frac{3C(n,2) - 1 - [C(n,2) - 1]\alpha}{(3-\alpha)C(n,2)} - \frac{3[C(n,2) - 1] - [C(n,2) + 1]\alpha}{(3-\alpha)C(n,2)} |b_1| \right\} \subset f(\mathbb{U}).$$

Theorem 3.3 *$f \in T_{\mathcal{H}}(n, \alpha)$ if and only if f can be expressed as*

$$f_k(z) = \sum_{k=1}^{\infty} (Y_k h_k + \Upsilon_k g_k) \tag{3.1}$$

where $h_1(z) = z$

$$\begin{aligned} h_k(z) &= z - \frac{1-\alpha}{(2k - \alpha - 1)C(n,k)} z^k, \qquad (k = 2, 3, ...), \\ g_k(z) &= z + \frac{1-\alpha}{(2k + \alpha + 1)C(n,k)} \overline{z}^k, \qquad (k = 1, 2, ...), \\ &\sum_{k=1}^{\infty} (Y_k + \Upsilon_k) = 1, \end{aligned}$$

$Y_k \geq 0$ and $\Upsilon_k \geq 0$. In particular, the extreme points of $T_{\mathcal{H}}(n, \alpha)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. Note that for f_k of the form (3.1), we may write

$$\begin{aligned} f_k(z) &= \sum_{k=1}^{\infty} (Y_k h_k + \Upsilon_k g_k) \\ &= \sum_{k=1}^{\infty} (Y_k + \Upsilon_k) z - \sum_{k=2}^{\infty} \frac{1-\alpha}{(2k - \alpha - 1)C(n,k)} Y_k z^k + \sum_{k=1}^{\infty} \frac{1-\alpha}{(2k + \alpha + 1)C(n,k)} \Upsilon_k \overline{z}^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{(2k - \alpha - 1)C(n,k)}{1-\alpha} \left(\frac{1-\alpha}{(2k - \alpha - 1)C(n,k)} Y_k \right) \\ &+ \sum_{k=1}^{\infty} \frac{(2k - \alpha - 1)C(n,k)}{1-\alpha} \left(\frac{1-\alpha}{(2k + \alpha + 1)C(n,k)} \Upsilon_k \right) = \sum_{k=2}^{\infty} Y_k + \sum_{k=1}^{\infty} \Upsilon_k \\ &\qquad = 1 - Y_1 \leq 1. \end{aligned}$$

Now the first part of the proof is complete, since by Theorem 2.2

$$\sum_{k=1}^{\infty}\left[\frac{(2k-\alpha-1)c(n,k)}{1-\alpha}\left(\frac{1-\alpha}{(2k-\alpha-1)c(n,k)}Y_k\right)\right.$$
$$\left.+\frac{(2k+\alpha+1)c(n,k)}{1-\alpha}\left(\frac{1-\alpha}{(2k+\alpha+1)c(n,k)}\Upsilon_k\right)\right]=\sum_{k=1}^{\infty}(Y_k+\Upsilon_k)=1$$

Conversely, suppose that $f\in T_{\mathcal{H}}(n,\alpha)$. Then

$$\sum_{k=1}^{\infty}\left[\frac{(2k-\alpha-1)C(n,k)}{1-\alpha}|a_k|+\frac{(2k+\alpha+1)C(n,k)}{1-\alpha}|b_k|\right]\leq 2.$$

Setting

$$Y_k=\frac{(2k-\alpha-1)c(n,k)}{1-\alpha}|a_k|,\ \Upsilon_k=\frac{(2k+\alpha+1)c(n,k)}{1-\alpha}|b_k|,\ 0\leq Y_k\leq 1,\ (k=1,2,3,...),$$

we obtain

$$f_k(z)=\sum_{k=1}^{\infty}(Y_kh_k+\Upsilon_kg_k) \text{ as required.}$$

4 Convolution and Convex Combination.

In this section, we show that the class $T_{\mathcal{H}}(n,\alpha)$ is invariant under convolution and convex combination of its member.

For harmonic functions

$$f(z)=z-\sum_{k=2}^{\infty}|a_k|z^k+\sum_{k=1}^{\infty}|b_k|\overline{z}^k$$

and

$$F(z)=z-\sum_{k=2}^{\infty}|A_k|z^k+\sum_{k=1}^{\infty}|B_k|\overline{z}^k$$

the convolution of f and F is given by

$$(f*F)(z)=f(z)*F(z)=z-\sum_{k=2}^{\infty}|a_k||A_k|z^k+\sum_{k=1}^{\infty}|b_k||B_k|\overline{z}^k \qquad (4.1)$$

Theorem 4.1 *For* $0 \leq \beta \leq \alpha < 1$ *let* $f \in T_{\mathcal{H}}(n,\alpha)$ *and* $F \in T_{\mathcal{H}}(n,\beta)$. *Then* $f * F \in T_{\mathcal{H}}(n,\alpha) \subset T_{\mathcal{H}}(n,\beta)$.

Proof. Suppose f and F are harmonic so that $f * F$ is given by above convolution. Since $f \in T_{\mathcal{H}}(n,\alpha)$ and $F \in T_{\mathcal{H}}(n,\beta)$, the coefficient of f and F must satisfy conditions given by Theorem 2.2. So for the coefficient of $f * F$ we can write

$$\begin{aligned}\sum_{k=2}^{\infty} \frac{(2k-\beta-1)C(n,k)}{1-\beta}|a_k||A_k| &+ \sum_{k=1}^{\infty} \frac{(2k+\beta+1)C(n,k)}{1-\beta}|b_k||B_k| \\ &\leq \sum_{k=2}^{\infty} \frac{(2k-\beta-1)C(n,k)}{1-\beta}|a_k| + \sum_{k=1}^{\infty} \frac{(2k+\beta+1)C(n,k)}{1-\beta}|b_k| \\ &\leq \sum_{k=2}^{\infty} \frac{(2k-\alpha-1)C(n,k)}{1-\alpha}|a_k| + \sum_{k=1}^{\infty} \frac{(2k+\alpha+1)C(n,k)}{1-\alpha}|b_k| \leq 1.\end{aligned}$$

Since $0 \leq \beta \leq \alpha < 1$ and $f \in T_{\mathcal{H}}(n,\alpha)$. Therefore $f * F \in T_{\mathcal{H}}(n,\alpha) \subset T_{\mathcal{H}}(n,\beta)$.

Now, we examine the convex combination of $T_{\mathcal{H}}(n,\alpha)$.

Let the functions $f_j(z)$ be defined , for $j = 1, 2, ..., m$ by

$$f_j(z) = z - \sum_{k=2}^{\infty} |a_{k,j}|z^k + \sum_{k=1}^{\infty} |b_{k,j}|\overline{z}^k. \tag{4.2}$$

Theorem 4.2 *Let the functions* $f_j(z)$ *defined by (4.2) be in the class* $T_{\mathcal{H}}(n,\alpha)$ *for every* $j = 1, 2, ..., m$. *Then the functions* $\Psi(z)$ *defined by*

$$\Psi(z) = \sum_{j=1}^{m} t_j f_j(z), \quad (t_j \geq 0), \tag{4.3}$$

is also in the class $T_{\mathcal{H}}(n,\alpha)$, *where* $\sum_{j=1}^{m} t_j = 1$.

Proof. According to the definition of Ψ, we can write

$$\Psi(z) = z - \sum_{k=2}^{\infty} \left(\sum_{j=1}^{m} t_j |a_{k,j}| \right) z^k + \sum_{k=1}^{\infty} \left(\sum_{j=1}^{m} t_j |b_{k,j}| \right) \overline{z}^k. \tag{4.4}$$

Further, since functions $f_j(z)$ are in $T_{\mathcal{H}}(n,\alpha)$ for every $j = 1, 2, ..., m$ we get

$$\sum_{k=2}^{\infty}\frac{(2k-\alpha-1)C(n,k)}{1-\alpha}|a_{k,j}|+\sum_{k=1}^{\infty}\frac{(2k+\alpha+1)C(n,k)}{1-\alpha}|b_{k,j}|)\leq 1, \tag{4.5}$$

for every $j=1,2,...,m$. We can see that

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{(2k-\alpha-1)C(n,k)}{1-\alpha}\Big(\sum_{j=1}^{m}t_j|a_{k,j}|\Big)+\sum_{k=1}^{\infty}\frac{(2k+\alpha+1)C(n,k)}{1-\alpha}\Big(\sum_{j=1}^{m}t_j|b_{k,j}|\Big)\\
=\ &\sum_{j=1}^{m}t_j\left(\sum_{k=2}^{\infty}\frac{(2k-\alpha-1)C(n,k)}{1-\alpha}|a_{k,j}|+\sum_{k=1}^{\infty}\frac{(2k+\alpha+1)C(n,k)}{1-\alpha}|b_{k,j}|)\right)\\
\leq\ &\sum_{j=1}^{m}t_j=1,
\end{aligned}$$

by Theorem 2.2, we have $\Psi(z)\in T_{\mathcal{H}}(n,\alpha)$.

Corollary 4.3 *The class $T_{\mathcal{H}}(n,\alpha)$ is close under convex linear combination.*

Proof. Let the functions $f_j(z)$ $(j=1,2)$ defined by (4.1) be in the class $T_{\mathcal{H}}(n,\alpha)$. Then the function $\Psi(z)$ defined by

$$\Psi(z)=\mu f_1(z)+(1-\mu)f_2(z) \qquad (0\leq\mu\leq 1)$$

is in the class $T_{\mathcal{H}}(n,\alpha)$. Also, by taking $m=2$, $t_1=\mu$ and $t_2=(1-\mu)$ in Theorem 4.1, we have the corollary.

5 Integral operator

Finally, we study the properties of an integral operator.

Theorem 5.1 *Let $f\in T_{\mathcal{H}}(n,\alpha)$ and let c be a real number such that $c>-1$. Then the function F defined by*

$$F(z)=\frac{c+1}{z^c}\int_o^z t^{c-1}f(t)dt,$$

belongs to the class $T_{\mathcal{H}}(n,\alpha)$.

Proof. From the representation of F, it follows that

$$\begin{aligned} F(z) &= \frac{c+1}{z^c}\int_o^z t^{c-1}\{h(t)+\overline{g(t)}\}dt \\ &= \frac{c+1}{z^c}\left(\int_o^z t^{c-1}h(t)dt + \overline{\int_o^z t^{c-1}g(t)dt}\right) \\ &= \frac{c+1}{z^c}\left(\int_o^z t^{c-1}\Big(z-\sum_{k=2}^{\infty}|a_k|t^k\Big)dt - \overline{\int_o^z t^{c-1}\Big(\sum_{k=1}^{\infty}|b_k|t^k\Big)dt}\right) \\ &= z-\sum_{k=1}^{\infty}|A_k|z^k - \sum_{k=1}^{\infty}|B_k|\overline{z}^k, \end{aligned}$$

where $|A_k| = \frac{c+1}{c+k}a_k$, $|B_k| = \frac{c+1}{c+k}|b_k|$. Therefore,

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{(2k-\alpha-1)C(n,k)}{1-\alpha}|A_k| + \sum_{k=1}^{\infty}\frac{(2k-\alpha-1)C(n,k)}{1-\alpha}|B_k| \\ =\ &\sum_{k=2}^{\infty}\frac{(2k-\alpha-1)C(n,k)}{1-\alpha}\frac{c+1}{c+k}|a_k| + \sum_{k=1}^{\infty}\frac{(2k-\alpha-1)C(n,k)}{1-\alpha}\frac{c+1}{c+k}|b_k| \\ \le\ &1. \end{aligned}$$

Since $f \in T_{\mathcal{H}}(n,\alpha)$, therefore by Theorem 2.2, $F \in T_{\mathcal{H}}(n,\alpha)$.

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