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High Degree Special Ruled Surfaces on Curves with General Moduli

E. Ballico[1]

Dept. of Mathematics
University of Trento
38050 Povo (TN), Italy
ballico@science.unitn.it

Abstract. Fix integers $g \geq 3$ and $d \geq 6g-4$. Here we describe the irreducible components of the set of all triples (C, E, V), where C is a smooth genus g curve with general moduli, E is a rank 2 vector bundle on C with degree d and V is a linear subspace of $H^0(C, E)$ such that $\dim(V) = d+2-2g$, V spans E and the morphism $\mathbb{P}(E) \to \mathbb{P}^{d-2g+1}$ induced by V is birational onto its image. For another proof (and more) see arXiv:math/0809.0373.

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1. Introduction

As in [2], Th. 1.2, for all integers d, g such that $g \geq 0$ and $d \geq 2g+2$ let $\mathcal{H}_{d,g}$ denote the unique irreducible component of the Hilbert scheme of surface scrolls of degree d and sectional genus g in $\mathbb{P}^{d-2g+1}$ whose general member represents a non-special linearly normal smooth scroll and which maps dominantly on the moduli space $\mathcal{M}_g$ of genus g smooth curve (with obvious modifications when $g = 1$). $\dim(\mathcal{H}_{d,g}) = (d-2g+2)^2 + 7(g-1)$. Moreover $\mathcal{H}_{d,g}$ is the unique component of the set of non-degenerate and non-special scrolls of degree d and genus g in $\mathbb{P}^{d-2g+1}$ ([2]). If $g \geq 2$, then a general element of $\mathcal{H}_{d,g}$ is associated to a general pair (C, E) where C is general in $\mathcal{M}_g$ and E is a general rank 2 stable vector bundle on C with degree d (and the converse holds) [3], Th. 5.4). Let $H[d, g]$ denote the reduction of the Hilbert scheme of all non-degenerate surface scrolls in $\mathbb{P}^{d-2g+1}$ with degree d and with sectional genus g. If $g \geq 2$ let $H'[d, g]$ denote the union of the irreducible components of $H[d, g]$ which dominate $\mathcal{M}_g$. If we drop the non-speciality assumption (or, equivalently, by Riemann-Roch the linearly normal condition),

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then other components which dominate $\mathcal{M}_g$ exist, at least if $d \geq 2g+11$ ([2], Example 5.12). For any smooth genus g curve let $H[d,C]$ denote the moduli space of degree d surface scrolls $S \subset \mathbb{P}^{d-2g+1}$ such that C is the normalization of a general hyperplane section of S. There are two interesting algebraic sets which deserve to be studied:

(a) The set $\mathcal{S}_{d,g}$ of all triples (C,E,V), where $C \in \mathcal{M}_g$, E is a rank 2 vector bundle on C, V is a $(d+2-2g)$-dimensional linear subspace of $H^0(C,E)$ spanning V and such that the morphism $\mathbb{P}(E) \to \mathbb{P}^{d+1-2g}$ is birational onto its image, up to isomorphisms of triples.

(b) The set $H[d,g]$.

We write $\widetilde{H}'[d,g]$ for the open subset of $\mathcal{S}_{d,g}$ obtained restricting the curve C to be with general moduli. Moreover, if we fix the curve $C \in \mathcal{M}_g$, we also get the set $\widetilde{H}[d,C]$ associated to the subset of $\mathcal{S}_{d,g}$ with C as the base curve. We write $\widetilde{\mathcal{H}_{d,g}}_{d,g}$ for the subset of $\mathcal{S}_{d,g}$ corresponding to non-special vector bundles. We write $\widetilde{H}[d,C]$ for the set of non-special degree d rank 2 vector bundles over the fixed curve C. There is a surjection ϕ between each set with a $\widetilde{}$ onto the corresponding set without the $\widetilde{}$. The fiber of any of these surjections over a scroll S corresponds to the subset of $\mathrm{Aut}(\mathbb{P}^{d+1-2g})$ inducing an automorphism of S.

We believe that the algebraic sets $\mathcal{S}_{d,g}$ deserve to be studied for their own sake. Anyway, their study is a preliminary step for the study of $\mathcal{H}_{d,g}$.

In this note we describe the irreducible components of $\widetilde{H}'[d,g]$ and of $\widetilde{H}[d,C]$ when $d \geq 6g-6$ and C is general (see Theorem 1). This result (and much more) was also proved in [4].

For all integers g,r,d let $\rho(g,r,d) := (r+1)d - rg - r(r+1)$ denote (the Brill-Noether number).

Theorem 1. *Fix an integer $g \geq 2$, an integer $d \geq 6g-4$ and general $C \in \mathcal{M}_g$. Let Θ be the set of all pairs (m,α), where $\lfloor (g+3)/3 \rfloor \leq m \leq 2g-2$, $\alpha > 0$, $m+\alpha > g$ and $\rho(g,m-g+\alpha,m) \geq 0$. For all integers r,m such that $r>0$, $\lfloor (g+3)/3 \rfloor \leq m \leq g$, and $\rho(g,r,m) = 0$ let $\tau(g,r,m)$ denote the number of all g^r_m on C. Let Θ' be the set of all triples (m,α,t), where $(m,\alpha) \in \Theta$, $t=1$ if $\rho(g,m-g+\alpha,m) > 0$, while t is any positive integer $\leq \tau(g,r,m)$ if $\rho(g,m-g+\alpha,m) = 0$. Set $\theta_g := \sharp(\Theta)$ and $\theta'_g := \sharp(\Theta')$.*

(i) *$\widetilde{H}'[d,g]$ has $\theta_g + 2$ irreducible components. $\widetilde{\mathcal{H}}_{d,g}$ is the only component of $H'[d,g]$ whose general member is associated to an indecomposable vector bundle. One irreducible component of $\widetilde{H}'[d,g]$ is associated to cones, i.e. its general member is associated to a triple $(C', \mathcal{O}_{C'} \oplus L, V)$, with C' general in $\mathcal{M}_g$, $L \in Pic^d(C')$ and V a general $(d+2-2g)$-dimensional linear subspace of $H^0(C', \mathcal{O}_{C'} \oplus L)$. The general member of the irreducible component with label $(m,\alpha) \in \Theta$ is of the form $(C', L \oplus M, V)$, with C' general in $\mathcal{M}_g$, $L \in Pic^{(d-m)}(C')$, $M \in W^{m+\alpha-g}_m(C')$ and V a general $(d+2-2g)$-dimensional linear subspace of $H^0(C', L \oplus M)$. The* *irreducible components of* $\widetilde{H}'[d,g]$ *whose general member has smooth image in* $\mathbb{P}^{d+1-2g}$ *are* $\widetilde{\mathcal{H}}_{d,g}$ *and the ones associated to a pair* (m,α) *with* $m-g+\alpha \geq 3$.

(ii) $\widetilde{H}[d,C]$ *has* θ'_g+2 *irreducible components whose description is as in part (i) taking* $C' := C$ *and* Θ' *instead of* Θ.

We work over an algebraically closed field $\mathbb{K}$. The Brill-Noether theory of special divisors on a curve with general moduli is true in arbitrary characteristic ([6]). For part (i) of Theorem 1 we assume $\mathrm{char}(\mathbb{K}) = 0$, because we quote a special case of [5]. B. Osserman exteded Eisenbud-Harris limit linear series to the positive characteristic case ([7]), allowing the interested reader to extend that part of [5] to the case $\mathrm{char}(\mathbb{K}) > 2g-2$.

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2. The proof

Fix an integer $g \geq 2$. A general member of $\mathcal{H}_{d,g}$ is associated to a pair (C,E) with C general in $\mathcal{M}_g$ and E a general degree d rank 2 stable vector bundle on C ([1], §2, [3], Th. 5.4). In particular the general $S \in \mathcal{H}_{d,g}$ is linearly normal. Let E be a rank two vector bundle on the smooth genus g curve X. Set $s(E) := \deg(E) - 2\deg(L)$, where L is a maximal degree line subsheaf of E. A classical theorem of C. Segre and M. Nagata says that $s(E) \leq g$ for all E. Take any maximal degree line subsheaf L of E. The maximality of the integer $\deg(L)$ gives that L is saturated in E, i.e. $E/L \in \mathrm{Pic}(X)$. Since $\deg(Hom(E/L,L)) = -s(E)$, $E \cong E/L \oplus L$ if $s(E) < 2-2g$. It is easy to check that any integer $2-2g \leq s(E) \leq g$ is realized by some indecomposable rank two vector bundle on X. The definition of stability (resp. semistability) gives that E is stable (resp. semistable, resp. properly semistable) if and only if $s(E) > 0$ (resp. $s(E) \geq 0$, resp. $s(E) = 0$). Since E is an extension of E/L by L and $s(E) = \deg(E/L) - \deg(E)$, then $s(E) \equiv \deg(E) \pmod 2$. Fix any irreducible component Γ of $\widetilde{H}[d,g]$ or of $H[d,g]$ or of $\widetilde{H}[d,C]$ or of $H[d,C]$ and take a general $S \in \Gamma$, say associated to the pair (C,E). Set $s(\Gamma) := s(E)$. A semicontinuity theorem for the integer $s(E)$ gives that $s(\Gamma)$ is well-defined. We have $s(E) \leq g$ and $s(E) \equiv d \pmod 2$. We have $s(\mathcal{H}_{d,g}) = g$ if $d \equiv g \pmod 2$ and $s(\mathcal{H}_{d,g}) = g-1$ if $d \equiv g-1 \pmod 2$.

Remark 1. Let C be a smooth genus g curve and E a rank 2 vector bundle on C. Set $d := \deg(E)$ and $s := s(E)$. Notice that $d \equiv s \pmod 2$. The integer s is often called the degree of stability of E. Let L be a maximal degree line subbundle of E. Hence $E/L \in \mathrm{Pic}(C)$, $\deg(L) = (d-s)/2$, $\deg(E/L) = (d+s)/2$ and E is an extension of E/L by L. Thus $h^1(C,E) = 0$ if $(d-|s|)/2 \geq 2g-1$, i.e. if $d \geq 4g-2+|s|$. Since $s(F) \geq 2-2g$ for any indecomposable rank 2 vector bundle F on C, we get $h^1(C,F) = 0$ for any indecomposable rank 2 vector bundle on C such that $\deg(F) \geq 6g-4$. If $g = 1$ it is sufficient to assume $\deg(F) \geq 1$. Thus if $g \geq 2$, $d \geq 6g-4$, $C \in \mathcal{M}_g$ and Γ is any component of $H[d,C]$, then either Γ is in the closure of the fiber over C of the map $\mathcal{H}_{d,g} \to \mathcal{M}_g$ or its general member is associated to a decomposable vector bundle. A similar statement follows for the irreducible components of $H[d,g]$, $\widetilde{H}[d,C]$ and $\widetilde{H}[d,g]$.

Remark 2. Fix an irreducible component Γ of $\widetilde{H}[d,g]$ or of $H[d,g]$ whose general element S is associated to a triple (C,E,V) with $E \cong L \oplus M$, $V \subseteq H^0(C,E)$, $L, M \in \mathrm{Pic}(C)$, $h^1(C,L) = 0$ and $h^1(C,M) > 0$. Set $m := \deg(M)$ and $r := h^0(C,M)-1$. The generality of S implies that L is a general element of $\mathrm{Pic}^{d-m}(C)$. The semicontinuity theorem for the degree of stability shows that $s(E') \leq 2m-d$ for all pairs (C',E') associated to some element of Γ. If $M \cong \mathcal{O}_C$, then S is a cone. From now on we assume that S is not a cone. Hence $m \geq \mathrm{gon}(C)$. Now assume that C has general moduli. Hence $m \geq \lfloor (g+3)/2 \rfloor$ and $\rho(g,r,m) \geq 0$. If $\rho(g,r,m) > 0$, the irreducibility of $G^r_d(C)$ and the fact that we may take as L any general element of $\mathrm{Pic}^{d-m}(C)$ gives that Γ contains all $(C, M \oplus L)$ with (M,L) general in $W^r_m(C) \times \mathrm{Pic}^{d-m}(C)$. If $\rho(g,r,d) = 0$, then the same is true (moving $C \in \mathcal{M}_g$) by [5]. If $\rho(g,r,m) > 0$, then even the fiber Γ_C at S of the fiber of the rational map from Γ into $\mathcal{M}_g$ contains all scrolls associated to vector bundles $L \oplus M$ with (L,M) varying in a non-empty open subset of $W^r_m(C) \times \mathrm{Pic}^{d-m}(C)$.

Remark 3. Fix integers g, m, α, d, such that $g \geq 2$, $\alpha > 0$, $m+\alpha-g \geq 1$, $d-m \geq 2g$, and $\rho(g, m+\alpha-g, m) \geq 0$. Fix a general genus g curve C. Let A (resp. B) be the set of all decomposable vector bundles $E = L \oplus M$ with $\deg(L) = d-m$, $\deg(M) = m$, M spanned and $h^1(C,M) = \alpha$ (resp. $\deg(L) = d-m-1$, $\deg(M) = m+1$, M spanned and $h^1(C,M) = \alpha-1$).

Since C is general, Brill-Noether theory gives that the set A' (resp. B') of all line bundles M appearing in the definition of A (resp. B) is non-empty, that it has pure dimension $\rho(g,m+\alpha-g,m)$ (resp. $\rho(g,m+\alpha-g,m+1) = \rho(g,m+\alpha-g,m)+m+\alpha-g$) and that it is irreducible if $\rho(g,m+\alpha-g,m) > 0$ (resp. $\rho(g,m+\alpha-g,m+1) > 0$). Notice that $h^0(C, L \oplus M) = d+2-2g+\alpha$. Let $G(d+2-2g, d+2-2g+\alpha)$ denote the Grassmannian of all $(d+2-2g)$-dimensional linear subspaces of $\mathbb{K}^{\oplus(d+2-2g+\alpha)}$. $G(d+2-2g, d+2-2g+\alpha)$ is irreducible and $\dim(G(d+2-2g, d+2-2g+\alpha)) = \alpha(d+2-2g)$. Notice that $\dim(G(d+2-2g, d+2-2g+\alpha-1)) = (\alpha-1)(d+2-2g)$. Let A'' (resp. B'') be the set of all pairs (E,V) with $E \in A$ (resp. $E \in B$) and V a $(d+2-2g)$-dimensional linear subspace of $H^0(C,E)$. Hence $\dim(A'') = \dim(B'') + d+2-2g-(m+\alpha-g) > \dim(B'')$. Hence A'' is not in the closure of B'' and the same is true for the set of all scrolls in $H[d,g]$ coming from A'' and B''. The same is true for all curves C with general moduli, i.e. for the subsets A_1 and B_1 of $\widetilde{H}[d,g]$ obtained from A'' and B'' varying C among a non-empty open subset of $\mathcal{M}_g$. Since $s(E) = 2m-d < 2m-d+2 = s(F)$ for all $(E,F) \in A \oplus B$, B_1 is disjoint from the closure of A_1 in $H[d,g]$.

Proof of Theorem 1. We first consider part (ii). For all $(m,\alpha,x)\in\Theta'$ let $\Gamma_{m,\alpha,x}$ denote the irreducible algebraic subset of $\widetilde{H}[d,C]$ parametrized by the pairs $(L\oplus M,V)$ with $L\oplus M$ labelled by (m,α,x) and V a general $(d+2-2g)$-dimensional linear subspace of $H^0(C,L\oplus M)$. Let Γ be an irreducible component of $\widetilde{H}[d,C]$ different from the fiber over C of the map $\widetilde{\mathcal{H}}_{d,g}\to\mathcal{M}_g$. Let (E,V) be a general element of Γ. Since $d\ge 6g-4$, $E\cong L\oplus M$, with, say $\deg(M)<\deg(L)$. Set $m:=\deg(M)<\deg(L)$. The induced scroll S is a cone if and only if $M\cong\mathcal{O}_C$, i.e. if and only if $m=0$. Assume $m>0$. Since M is spanned, Γ is the closure of a unique $\Gamma_{m,\alpha,x}$ with $\alpha:=h^1(C,M)$ (Remark 2). To prove part (ii), except the smoothness assertion, it is sufficient to prove that Γ contains no $\Gamma_{m',\alpha',x'}$ with $(m',\alpha,x')\ne(m,\alpha,x)$. Assume that this is not the case, and take $(m',\alpha,x')\ne(m,\alpha,x)$ such that $\Gamma_{\alpha',m',x'}\subset\Gamma$. Look at Remark 3. If $\rho(g,m_1+\alpha_1-g,m_1)=0$, then all $\Gamma_{m_1,\alpha_1,t}$, $1\le t\le\tau(g,m_1+\alpha_1-g,m_1)$, have the same dimension. Hence $(m',\alpha')\ne(m,\alpha)$. The semicontinuity theorem for cohomology gives $\alpha'\ge\alpha$. The semicontinuity theorem for the stability degree $s(E)$ gives $m'\le m$. We have $\dim(\Gamma_{m,\alpha,t})-\dim(\Gamma_{m',\alpha',t'})=\rho(g,m+\alpha-g,\alpha)+\alpha(d+2-2g)-\rho(g,m'+\alpha'-g,\alpha)+\alpha'(d+2-2g)$. Since $\dim(\Gamma_{m,\alpha,t})>\dim(\Gamma_{m',\alpha',t'})$ and $m'\le m$, we easily get $\alpha'>\alpha$. Since $d+2-2g>g$, while $|\rho(g,m+\alpha-g,\alpha))-\rho(g,m'+\alpha'-g,\alpha)|\le g$, we get $\dim(\Gamma_{m,\alpha,t})<\dim(\Gamma_{m',\alpha',t'})$, contradiction. Now we check that last assertion of (i) and (ii). L is very ample. Since M is general in $G^{m+\alpha-g}_m(C)$, M is very ample if and only if $m+\alpha-g\ge 3$. Since $d+2-2g\ge 5$, we get that if $m+\alpha-g\ge 3$, then $S\cong\mathbb{P}(L\oplus M)$ and hence S is smooth. Now assume $m+\alpha-g\le 2$ and take a general $S\in\Gamma_{m,\alpha,t}$, say represented by a pair $(L\oplus M,V)$. M is spanned, but not very ample. The generality of S implies the generality of M in $W^{m+\alpha-g}_m$ when $\rho(g,g+\alpha-m,m)>0$. Hence there are $P,Q\in C$, such that $P\ne Q$ and $h^0(C,M(-P-Q))=h^0(C,M)-1$. Hence $h^0(C,(L\oplus M)(-P-Q))=h^0(C,L\oplus M)-3$. Let $S_1\subset\mathbb{P}^{d+1-g+\alpha}$ denote the image of $\mathbb{P}(L\oplus M)$ obtained using $H^0(C,L\oplus M)$. Since $h^0(C,(L\oplus M)(-P-Q))=h^0(C,L\oplus M)-3$, the fibers D_P and D_Q of the ruling of $\mathbb{P}(L\oplus M)$ over P and over Q are coplanar. Hence $D_P\cap D_Q\ne\emptyset$. Hence S_1 is not smooth. Hence a general projection of S_1 in $\mathbb{P}^{d+1-2g}$ is not smooth. Hence a general element of $\Gamma_{m,\alpha,t}$ is not smooth, concluding the proof of part (ii). Part (i) follows from part (ii), the fact that $G^x_y(C')$ and $W^x_y(C')$ are irreducible and non-empty for all general C' and all x,y such that $\rho(g,x,y)>0$ and the irreducibility statement for W^x_y and G^x_y over a dense open subset of $\mathcal{M}_g$ proved in [5] in the case $\rho(g,x,y)=0$. □

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