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Estimation Based on Progressively Censored Data from the Burr Model

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Abstract

Based on progressively Type-II censored samples, the uniformly minimum variance unbiased (UMVU), Bayes and empirical Bayes estimates for the unknown parameter and the reliability function of the Burr model are derived. The Bayes and empirical Bayes estimates are obtained based on absolute error and logarithmic loss functions. We also present a numerical example and a Monte Carlo simulation study to illustrate the results.

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1 Introduction

The two parameter Burr type XII distribution (denoted by Burr(c, θ)) was first introduced in literature by Burr (1942). The probability density function (pdf), cumulative distribution function (cdf) and reliability function (at some t) of the Burr(c, θ) distribution are given, respectively, by

$$f(x;\theta) = c\ \theta x^{c-1}(1+x^c)^{-(\theta+1)}, \quad x>0, \quad (\theta>0,\ c>0) \tag{1.1}$$

$$F(x;\theta) = 1-(1+x^c)^{-\theta}, \qquad x>0, \tag{1.2}$$

and

$$R(t) = (1+t^c)^{-\theta}. \tag{1.3}$$

Inference for Burr XII model based on complete and censored samples were discussed by many authors. Evans and Ragab (1983) obtained Bayes estimates of θ and the reliability function based on Type-II censored samples.

Ali Mousa (1995) obtained empirical Bayes estimation of the parameter θ and the reliability function based on accelerated Type-II censored data. Based on complete samples, Moore and Papadopoulos (2000) obtained Bayes estimates of θ and the reliability function when the parameter c is assumed to be known. A1i Mousa and and Jaheen (2002) obtained Bayes approximate estimates for the two parameters and reliability function of the Burr(c, θ) based on progressive Type-II censored samples. Based on the same progressive samples as above, Soliman (2005) obtained the Bayes estimates using both the symmetric (Squared Error) loss function, and asymmetric (LINEX, General Entropy) loss functions.

In this paper, based on progressively Type-II censored samples, we obtain the UMVU, Bayes and empirical Bayes estimates for the unknown parameter and the reliability function of the Burr(c, θ) when the parameter c is assumed to be known. A brief account for the progressive censoring will be given in Section 2. In Section 3, the UMVU estimates for the parameter θ and the reliability function are obtained. In Section 4, the Bayes and empirical Bayes estimates are obtained under two different loss functions, absolute difference and logarithmic. Finally in Section 5, a numerical example and a Monte Carlo simulation study are given to illustrate the results of estimation.

2 Progressive Type-II Censoring

The progressive Type-II censoring scheme can be described as follows: n units are put on life test at time 0. Immediately following the first failure, R_1 surviving units are removed from the test at random. Then, immediately following the second failure, R_2 surviving units are removed from the test at random. This process continues until, at the time of the m-th failure, all the remaining $R_m = n - R_1 - R_2 - ... - R_{m-1} - m$ units are removed from the experiment. The R_i's are fixed prior to study. If $R_1 = R_2 = ... = R_m = 0$, we have $n = m$ which corresponds to the complete sample situation. If $R_1 = R_2 = ... = R_{m-1} = 0$, then $R_m = n-m$ which corresponds to the conventional Type-II right censoring scheme. Let $X_{1:m:n}, \cdots, X_{m:m:n}$ be a progressively Type-II censored sample with $(R_1, \cdots, R_m)$ being the progressive censoring scheme. For simplicity of notation, we will use x_i instead of $X_{i:m:n}$, with $i = 1, ..., m$. The likelihood function based on this progressively Type-II censored sample is then (see Balakrishnan and Aggarwala(2000))

$$L(\theta|\mathbf{x}) = A \prod_{i=1}^{m} f(x_i, \theta)\{1 - F(x_i, \theta)\}^{R_i}, \tag{2.1}$$

where

$$A = n(n-1-R_1)(n-2-R_1-R_2) \ \cdots \ (n-m+1-R_1-\cdots-R_{m-1}),$$

and $\mathbf{x} = (x_1, ..., x_m)$.

When data are obtained by progressive censoring, inference problems for various models have been studied by many authors including Viveros and Balakrishnan (1994), Balakrishnan and Kannan (2000), Balakrishnan et al. (2003), Balakrishnan and Asgharzadeh (2005). For further details on progressively censoring and relevant references,the reader may refer to the book by Balakrishnan and Aggarwala (2000).

3 UMVU Estimates

Let $X_1 = X_{1:m:n}, \cdots, X_m = X_{m:m:n}$ be a progressively Type-II censored samples from a life test on items whose lifetimes have a Burr(c, θ) distribution (1.1). Substituting (1.1) and (1.2) into (2.1) the likelihood function is

$$f(\mathbf{x}|\theta) = A\theta^m c^m \prod_{i=1}^{m} x_i^{c-1}(1+x_i^c)^{-(R_i+1)\theta-1}. \tag{3.1}$$

From (3.1), the maximum likelihood (ML) estimates for θ and $R(t)$ are

$$\widehat{\theta}_{ML} = \frac{m}{S}, \quad \text{and} \quad \widehat{R}(t)_{ML} = (1+t^c)^{-\widehat{\theta}_{ML}}, \tag{3.2}$$

where $S = \sum_{i=1}^{m}(R_i+1)\ln(1+X_i^c)$.

Let us now consider the UMVU estimators of θ and $R(t)$. We know that if $X \sim Burr(c,\theta)$, then $Y = 1 + X^c$ is distributed as $Pareto(\theta)$ with pdf

$$f(y;\theta) = \theta y^{-\theta-1}, \quad y > 1, \ \theta > 0.$$

Let $Y_{1:m:n}, \cdots, Y_{m:m:n}$ be a progressively Type-II censored sample from the $Pareto(\theta)$ distribution obtained from a sample of size n with the censoring scheme $(R_1, \cdots, R_m)$. Then, from a known result about the progressively Type-II censored sample from Pareto model (see Balakrishnan and Aggarwala (2000), Page 24), we have

$$Z_1 = Y_{1:m:n} \sim pareto(n\theta)$$

$$Z_i = \frac{Y_{i:m:n}}{Y_{i-1:m:n}} \sim pareto((n - \sum_{j=1}^{i-1}(R_j+1)\theta), \quad i = 2, ..., m$$

and the random variables $Z_{1:m:n}, \cdots, Z_{m:m:n}$ are all statistically independent. Now, since

$$U_1 = 2n\theta \ln(Z_1) \sim \chi^2_2$$

$$U_2 = 2\theta \sum_{i=2}^{m}(n - \sum_{j=1}^{i-1}(R_j+1))\ln(Z_i) \sim \chi^2_{2(m-1)},$$

we conclude that

$$U_1 + U_2 = 2\theta \sum_{i=1}^{m} (R_i + 1) \ln(1 + X_i^c) = 2\theta S \sim \chi^2(2m). \tag{3.3}$$

Here $\chi^2(p)$ denotes the central chi-square distribution with p degrees of freedom.

From the joint density of $X_{1:m:n}, \cdots, X_{m:m:n}$ in (3.1), it is easy to see that $S = \sum_{i=1}^{m} (R_i + 1) \ln(1 + X_i^c)$ is a complete sufficient statistics. By (3.3), since $E(S) = \theta/(n-1)$, hence the UMVU estimator of θ is

$$\widehat{\theta}_{UMVU} = \frac{n-1}{S}. \tag{3.4}$$

In order to derive the UMVU estimator of $R(t)$, define

$$g(x) = (\frac{x - \ln(1 + t^c)}{x})^{m-1}, \quad x > \ln(1 + t^c).$$

Then, it is easy to verify that $E[g(S)] = R(t)$. Hence, the UMVU estimator of $R(t)$ is

$$\widehat{R}(t)_{UMVU} = (\frac{S - \ln(1 + t^c)}{S})^{m-1}, \quad S > \ln(1 + t^c). \tag{3.5}$$

4 Bayes and Empirical Bayes Estimates

In Bayesian analysis, the loss function is used to represent a penalty associated with each of the the possible estimators. The usual loss function for carrying out a Bayesian analysis is the squared error (SE), or quadratic loss function. Such a choice is arbitrary and its popularity is due to its analytical tractability. But in life testing and reliability problems, the use of SE loss function may be inappropriate in some situations (see for example Basu and Ebrahimi (1991)). In this section, we consider two different loss functions, absolute difference and logarithmic.

We assume that c is known and derive the Bayes and empirical Bayes estimates for the parameter θ and the reliability of the Burr model under the above mentioned loss functions.

4.1 Bayes Estimates

Consider the family of gamma distributions, $\Gamma(1,\ \beta)$, with pdf

$$\pi(\theta|\beta) = \beta e^{-\beta\theta}, \quad \theta > 0, \quad \beta > 0 \tag{4.1}$$

as the conjugate family of prior distributions for θ. It follows from (3.1) and (4.1) that the posterior distribution of θ is

$$\pi(\theta|\mathbf{x}) = \frac{(\beta+S)^{m+1}}{\Gamma(m+1)}\theta^m e^{-(\beta+S)\theta}, \quad \theta > 0 \tag{4.2}$$

which is $\Gamma(m+1, \beta+S)$. The posterior given above is used to calculate the Bayes estimates.

The absolute difference loss function is

$$L_{AD}(\widehat{u}, u) = |\widehat{u} - u|,$$

where $\widehat{u}$ is an estimate of $u = u(\theta)$. The Bayes estimate of the parameter θ under this loss function is the median of the posterior distribution $\pi(\theta|\mathbf{x})$. Since the posterior distribution is gamma , we obtain the Bayes estimate for θ as

$$\widehat{\theta}_{AD} = \frac{\nu_{2(m+1)}}{2(\beta+S)}. \tag{4.3}$$

where $\nu_{2(m+1)}$ is the median of a chi-squared pdf with 2(m+1) degrees of freedom.

Now, we consider the Bayes estimate of $R(\theta,\ t)$ under absolute difference loss. Since $R(\theta,\ t)$ is strictly monotone then, the Bayes estimate of $R(\theta, t)$, is of the form $R(\widehat{\theta},\ t)$ where $\widehat{\theta}$ is the posterior median of θ. Thus, the Bayes estimate of $R(t)$ is

$$\widehat{R}(t)_{AD} = (1+t^c)^{-\widehat{\theta}_{AD}}. \tag{4.4}$$

The logarithmic loss function is

$$L_{LN}(\widehat{u}, u) = (\ln\frac{\widehat{u}}{u})^2 = (\ln\widehat{u} - \ln u)^2. \tag{4.5}$$

This loss function that has been introduced by Brown (1968), places a small weight on estimates whose ratios to the true value are close to one and proportionately more weight on estimates whose ration to the true value are significantly different from one. The posterior expectation of (4.5) is

$$E[L_{LN}(\widehat{u}, u)|\mathbf{x}] = (\ln\widehat{u})^2 - 2(\ln\widehat{u})\ E(\ln u|\mathbf{x}) + E[(\ln u)^2|\mathbf{x}]. \tag{4.6}$$

Under the logarithmic loss function (4.5), the Bayes estimate of u (denoted by $\widehat{u}_{LN}$) is the value of $\widehat{u}$ which minimizes (4.6). It is

$$\widehat{u}_{LN} = exp[E(\ln u|\mathbf{x})], \tag{4.7}$$

provided that $E(\ln u|\mathbf{x})$ exists, and is finite. So the Bayes estimator of θ is

$$\begin{aligned} \widehat{\theta}_{LN} &= exp[E(\ln\theta|\mathbf{x})] \\ &= exp[\int_0^\infty \ln\theta\ \frac{(\beta+S)^{m+1}}{\Gamma(m+1)}\theta^m e^{-(\beta+S)\theta} d\theta]. \end{aligned}$$

It can be shown that

$$E(\ln\theta|\mathbf{x}) = \Psi(m+1) - \ln(\beta+S),$$

where $\Psi(x)$ is the digamma function defined by $\Psi(x) = d/dx \ln\Gamma(x) = \Gamma'(x)/\Gamma(x)$. Hence, we have

$$\begin{aligned}\widehat{\theta}_{LN} &= exp[E(\ln\theta|\mathbf{x})]\\ &= exp[\Psi(m+1) - \ln(\beta+S)]\\ &= \frac{e^{\Psi(m+1)}}{\beta+S}. \end{aligned} \tag{4.8}$$

Also, after setting $u = R(t)$ in (4.7), the Bayes estimator of $R(t)$ is

$$\begin{aligned}\widehat{R}(t)_{LN} &= exp[E(\ln R(t)|\mathbf{x})]\\ &= exp[-\ln(1+t^c)E(\theta|\mathbf{x})]\\ &= (1+t^c)^{-\frac{m+1}{\beta+S}}. \end{aligned} \tag{4.9}$$

4.2 Empirical Bayes Estimates

In (4.3), (4.4), (4.8) and (4.9) the hyper-parameter β is an unknown constant, so the Bayes estimates can not be used directly. When the prior parameter β is unknown, we may use the empirical Bayes approach to get its estimate. From (3.1) and (4.1), we calculate the marginal pdf of $\mathbf{x}$, with density

$$\begin{aligned}m(\mathbf{x}|\beta) &= \int_0^\infty f(\mathbf{x}|\theta)\pi(\theta|\beta)d\theta\\ &= Ac^m\prod_{i=1}^m \frac{x_i^{c-1}}{1+x_i^c}\beta\int_0^\infty \theta^m e^{-(\beta+S)\theta}d\theta\\ &= Ac^m\prod_{i=1}^m \frac{x_i^{c-1}}{1+x_i^c}\frac{\beta\,\Gamma(m+1)}{(\beta+S)^{m+1}}.\end{aligned}$$

Based on $m(\mathbf{x}|\beta)$, we obtain an estimate, $\widehat{\beta}$, of β. The MlE of β is

$$\widehat{\beta} = \frac{S}{m}. \tag{4.10}$$

Applying this estimate in (4.3) and (4.4), the empirical Bayes estimates for the parameter θ and the reliability function under the absolute difference loss are

$$\widehat{\theta}_{EAD} = \frac{\nu_{2(m+1)}}{2(\widehat{\beta}+S)}, \text{ and } \quad \widehat{R}(t)_{EAD} = (1+t^c)^{-\widehat{\theta}_{EAD}}. \tag{4.11}$$

Similarly, the empirical Bayes estimates for the parameter θ and the reliability function under the logarithmic loss are

$$\widehat{\theta}_{ELN} = \frac{e^{\Psi(m+1)}}{\widehat{\beta}+S}, \text{ and } \quad \widehat{R}(t)_{ELN} = (1+t^c)^{-\frac{m+1}{\widehat{\beta}+S}}. \tag{4.12}$$

Table 1. Estimates of θ and $R(t)$(for $t = 1,\ 2$)

	Exact	ML	UMVU	AD	LN	EAD	ELN
$\widehat{\theta}$	0.5243	0.5415	0.4738	0.5168	0.5070	0.5215	0.5117
$\widehat{R}(t=1)$	0.6953	0.6871	0.7144	0.6989	0.6894	0.6966	0.6871
$\widehat{R}(t=2)$	0.3160	0.3043	0.3240	0.3213	0.3076	0.3179	0.3043

5 Numerical Computations

In this section, we present a numerical example and a Monte Carlo simulation study to illustrate the methods of inference developed in this paper.

5.1 Illustrative Example

The ML, UMVU, Bayes and empirical Bayes estimates of the parameter θ and reliability function are obtained according to the following steps:
1. For given value of $\beta = 2$, we generate $\theta = 0.5243$ from the prior pdf (4.1).
2. Using the value $\theta = 0.5243$ from step 1, we generate a progressively Type-II censored sample of size $m = 8$ with the censoring scheme $R = (0, 0, 4, 0, 3, 0, 0, 5)$ from the Burr(c=3, $\theta = 0.5243$) distribution, according to an algorithm due to Balakrishnan and Aggarwala (2000). This sample is

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0.1483  0.4074  0.4539  0.6083  0.6542   1.1700   1.3970  1.8210
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3. Using these data, the ML, UMVU and Bayes estimates of θ and the reliability function are computed from (3.2), (3.4), (3.5), (4.3), (4.4), (4.8) and (4.9). All results are displayed in Table 1.
4. The MLE $\widehat{\beta} = 1.8469$ is computed using (4.10).
5. Applying the estimate $\widehat{\beta} = 1.8469$ in the Bayes estimates, we also computed and reported the empirical Bayes estimates in Table 1.

5.2 Simulation Results

A simulation study was carried out to compare the performance of the different estimates. The steps are as follows in detail:
1. For given value of the prior parameter β, we generate θ from the prior pdf (4.1). Substituting θ in to (1.3), $R(t)$ can be obtained for given t.
2. For given $c, \theta,\ m,\ n$ and $(R_1, \cdots, R_m)$, the progressively Type-II censored sample $(x_1, \cdots, x_m)$ can be derived according to the algorithm of Balakrishna and Aggarwala (2000)
3. For given t, The ML and UMVU estimates of θ and $R(t)$ are calculated .
4. The different Bayes and empirical Bayes estimates of θ are computed
5. Steps 1-4 are repeated 5000 times, and the estimated risks (ER) of the different estimates are computed as the average of their squared deviations.

Table 2. Censoring scheme $(R_1, \cdots, R_m)$

Case	n	m	Scheme
1	20	10	(0,2,1,0,1,1,2,0,0,3)
2	30	20	(0,1,0,0,0,0,2,0,0,0,2,0,0,3,0,0,1,0,0,1)
3	40	30	(1,0,2,0,0,1,0,2,0,0,0,1,0,0,0,1,0,0,1,0,0,0,0,0,0,0,0,0,0,1)

Table 3. Estimated risks (ER) of the estimates of θ $(c = 3,\ \beta = 2)$.

Case	ML	UMVU	AD	LN	EAD	ELN
1	0.0459	0.0351	0.0292	0.0280	0.0419	0.0401
2	0.0170	0.0147	0.0140	0.0136	0.0161	0.0157
3	0.0108	0.0098	0.0095	0.0094	0.0104	0.0102

Tables 3 and 4 display the estimated risks of the ML, UMVU, Bayes and Empirical Bayes estimates of θ and $R(t = 1)$. Three different cases of the the sample size and censoring scheme are shown in Table 2.

6 Conclusion

1- Table 3 shows that the Bayes estimates relative to the logarithmic loss function have almost the smallest estimated risk (ER) as compared with the ML and UMVU estimates or absolute Bayes estimates.
2- From Tables 3 and 4, as the effective sample proportion m/n increases, the estimated risk of the estimates reduces significantly.
3- It is also observed that the empirical Bayes estimates are almost as efficient as the Bayes estimates when the sample size m/n is increasing.

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Table 4. Estimated risks (ER) of the estimates of $R(t)$ $(t = 1,\ c = 3,\ \beta = 2)$.

Case	ML	UMVU	AD	LN	EAD	ELN
1	0.0312	0.0387	0.0333	0.0309	0.0334	0.0312
2	0.0292	0.0326	0.0306	0.0293	0.0305	0.0292
3	0.0291	0.0314	0.0301	0.0292	0.0300	0.0291

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