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The $(\in, \in \vee q)$-Fuzzy Prime Ideals and Maximal Ideals of a Semigroup

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Abstract

In this paper, using the idea of quasi-coincidence of a fuzzy point with a fuzzy set, the concepts of $(\in, \in \vee q)$-fuzzy ideals, prime(resp.weak-prime, semiprime) ideals and maximal ideals of a semigroup are introduced, and the characterizations of them are given. Also, some related properties are investigated. Finally, in the sense of homomorphism between two crisp semigroups, the images and inverse images of $(\in, \in \vee q)$-fuzzy ideals, prime(resp.weak-prime, semiprime) ideals and maximal ideals are studied.

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1 Introduction

The concept of fuzzy set was introduced by Zadeh[1]. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, groupoids, real analysis, measure theory, topology, ect. Many notions of mathematics are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. It was first applied to the theory of groups by A.Rosenfeld[2]. A.Rosenfeld's definition was extended in C.V.Negoita and D.A.Ralescu[3] and redefined by S.K.Bhakat and P.Das[5]. Afterwards, many authors further introduced fuzzy subsemigroups, fuzzy subrings, fuzzy ideals and so on(see,e.g.,[9-13,16-18]). The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [14], played a vital role to generate some different types of fuzzy subgroups. It is worth pointing out that Bhakat and Das [5] gave the concept of (α,β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in,\in\vee q)$-fuzzy subgroup. In particular, $(\in,\in\vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. Also, Bhakat and Das [6,7] considered the $(\in,\in\vee q)$-fuzzy subgroup of a group and $(\in,\in\vee q)$-fuzzy ideals of a ring, respectively. Recently, Bhakat [8] studied the $(\in,\in\vee q)$-fuzzy normal, quasinormal and maximal subgroups. Y.B. Yun and S.Z. Song [9] introduced the concept of (α,β)-fuzzy interior ideals of a semigroup. As a further study, we will investigate the $(\in,\in\vee q)$-fuzzy ideals, prime(resp.weak-prime, semiprime) ideals and maximal ideals of a semigroup.

2 Preliminaries

In this section, we will briefly recall some basics notions that will be used in the sequel.

Let X be any non-empty set. A mapping $\mu : X \to [0,1]$ is called a fuzzy subset in X. For any $A \subseteq X$ and $r \in [0,1]$, $r_A : X \to [0,1]$ is defined by

$$r_A(x) = \begin{cases} r & if\ x \in A, \\ 0 & otherwise. \end{cases}$$

for all $x \in X$. In particular, if $r = 1$, then 1_A is said to be the characteristic function of A, and we shall use the symbol C_A for 1_A.

Defition 2.1(*Cf.* [14]).Let X be any non-empty set. A fuzzy subset μ in X defined by

$$\mu(y) = \begin{cases} r(\neq 0) & if\ y = x, \\ 0 & otherwise. \end{cases}$$

is said to be a fuzzy point with support x and value r and is denoted by x_r.

Defition 2.2(*Cf.* [14]) A fuzzy point x_r is said to belong to(resp.be quasi-coincident with) a fuzzy set μ, written as $x_r \in \mu$(resp.$x_r\, q\, \mu$) if$\mu(x) \geq r$ (resp.$\mu(x) + r > 1$); If $\mu(x) \geq r$ or $\mu(x) + r > 1$, then we write $x_r \in \vee q\ \mu$.

Defition 2.3 Let X be a non-empty set, μ and ν be fuzzy subsets of X. If $x_r \in \nu$ implies $x_r \in \vee q\ \mu$, then we write $\nu \subseteq \vee q\ \mu$.

Defition 2.4 Let X be any non-empty set and μ a fuzzy subset in X. For all $r \in (0,1]$, Denote the sets $\mu_r = \{x \in X | \mu(x) \geq r\}$(resp.$\mu_r^s = \{x \in X | \mu(x) > r\}$), $[\mu]_r = \{x \in X | x_r \in \vee q\, \mu\}$ and $Supp(\mu) = \{x \in X | \mu(x) > 0\}$, which are called $r-$levelset(resp.$r-$strong levelset), $r- \in \vee q$ set and supporting set, respectively.

Defition 2.5(*Cf.* [10, 11]) Let S be a semigroup, μ and ν be fuzzy subsets in S. Then the product of μ and ν denoted by $\mu \circ \nu$ is defined by

$$(\mu \circ \nu)(x) = \begin{cases} \bigvee\limits_{x=yz} \mu(y) \wedge \nu(z) & for \;\; y, z \in S, \;\; x = yz \\ 0 & otherwise, \end{cases}$$

for all $x \in S$.

Clearly, for any fuzzy points x_r and y_t in S, $x_r \circ y_t = (xy)_{r \wedge t}$. Also, for any fuzzy subsets μ, ν and ω in S, $(\mu \circ \nu) \circ \omega = \mu \circ (\nu \circ \omega)$.

Defition 2.6(*Cf.* [10, 11]) Let S be a semigroup. A fuzzy subset μ in S is called a fuzzy subsemigroup of S if

$$\mu(xy) \geq \mu(x) \wedge \mu(y) \;\; \forall x, y \in S.$$

A fuzzy subset μ in S is called a fuzzy ideal of S if

$$\mu(xy) \geq \mu(x) \vee \mu(y) \;\; \forall x, y \in S.$$

3 The $(\in, \in \vee q)$-fuzzy ideals of a semigroup

In the sequel, unless otherwise stated, S always represents any given semigroup. For any $r, t \in [0,1]$, $M(r,t)$ will denote $r \wedge t$. $\overline{\in \vee q}$ means $\in \vee q$ does not hold and $\overline{\subseteq \vee q}$ implies $\subseteq \vee q$ is not true.

Defition 3.1 A fuzzy subset μ in S is said to be a

(1) $(\in, \in \vee q)$-fuzzy subsemigroup, if $\forall x, y \in S, \; r, t \in (0,1]$, $x_r, y_t \in \mu \Rightarrow x_r \circ y_t \in \vee q\, \mu$;

(2) $(\in, \in \vee q)$-fuzzy ideal, if $\forall x, y \in S, \; r \in (0,1]$, $x_r \in \mu \Rightarrow (yx)_r \in \vee q\, \mu$,$(xy)_r \in \vee q \; \mu$.

We note that the semigroup S can be considered a fuzzy subset in itself and we write $S = C_S$(the characteristic function of S), i.e, $S(x) = 1$ for all $x \in S$.

Theorem 3.2 Let μ be any non-empty fuzzy subset in S. Then the following statements are equivalent:

(1) μ is an $(\in, \in \vee q)$-fuzzy ideal of S;

(2) $\forall x, y \in S$, $\mu(xy) \geq M(\mu(x) \vee \mu(y), 0.5)$;

(3) $S \circ \mu \cup \mu \circ S \subseteq \vee q \; \mu$;

(4) $(S \circ \mu \cup \mu \circ S) \cap 0.5_S \subseteq \mu$;

(5) $\forall r \in (0, 0.5]$, if μ_r(resp.μ_r^s) is non-empty, then μ_r(resp.μ_r^s) is an ideal of S;

(6) $\forall r \in (0, 0.5]$, if $[\mu]_r$ is non-empty, then $[\mu]_r$ is an ideal of S.

Proof. (1)$\Rightarrow$(2) Assume that (1) holds. For all $x, y \in S$, if possible, let $\mu(xy) < M(\mu(x) \vee \mu(y), 0.5)$. Choose r such that $\mu(xy) < r < M(\mu(x) \vee \mu(y), 0.5)$. Then $x_r \in \mu$ or $y_r \in \mu$ but $(xy)_r \overline{\in \vee q}\ \mu$, a contradiction. Therefore (1) implies (2).

(2)$\Rightarrow$(3) Assume that (2) holds. For all $x_r \in S \circ \mu \cup \mu \circ S$, if possible, let $x_r \overline{\in \vee q}\ \mu$. Then $\mu(x) < r$ and $\mu(x) + r \leq 1$. Hence $\mu(x) < 0.5$. Now, $r \leq (S \circ \mu \cup \mu \circ S)(x) = (S \circ \mu)(x) \vee (\mu \circ S)(x) = (\bigvee\limits_{x=yz} \mu(z)) \vee (\bigvee\limits_{x=yz} \mu(y)) = \bigvee\limits_{x=yz} \mu(y) \vee \mu(z) \leq \bigvee\limits_{x=yz} \mu(yz)$ *(since* $0.5 > \mu(x) = \mu(yz) \geq M(\mu(y) \vee \mu(z), 0.5) = \mu(y) \vee \mu(z)$*)* $= \mu(x)$, a contradiction. Hence, $x_r \in \vee q\ \mu$. This implies that $S \circ \mu \cup \mu \circ S \subseteq \vee q\ \mu$. Therefore (2) implies (3).

(3)$\Rightarrow$(4) Assume that (3) holds. For all $x_r \in (S \circ \mu \cup \mu \circ S) \cap 0.5_S$, namely $r \leq 0.5$ and $x_r \in S \circ \mu \cup \mu \circ S$. If $x_r \notin \mu$, then $\mu(x) < r \leq 0.5$, this implies that $x_r \overline{\in \vee q}\ \mu$, which contradicts $x_r \in S \circ \mu \cup \mu \circ S \subseteq \vee q\ \mu$. Hence $x_r \in \mu$. This implies that $(S \circ \mu \cup \mu \circ S) \cap 0.5_S \subseteq \mu$. Therefore (3) implies (4).

(4)$\Rightarrow$(5) Assume that (4) holds and let $r \in (0, 0.5]$ such that μ_r is non-empty. For all $x \in S, y \in \mu_r$, that is $\mu(y) \geq r$, we then have $\mu(xy) \geq ((S \circ \mu \cup \mu \circ S) \cap 0.5_S)(xy) = M((S \circ \mu \cup \mu \circ S)(xy), 0.5) \geq M((S \circ \mu)(xy), 0.5) = M(\bigvee\limits_{ab=xy} \mu(b), 0.5) \geq M(\mu(y), 0.5) \geq M(r, 0.5) = r$. This implies that $xy \in \mu_r$. Similarly, $yx \in \mu_r$. Hence μ_r is a ideal of S. The case for μ_r^s can be similarly disposed of. Therefore (4) implies (5).

(5)$\Rightarrow$(6) Assume that (5) holds let $r \in (0, 1]$ such that $[\mu]_r$ is non-empty. For all $x \in S, y \in [\mu]_r$, then $y_r \in \vee q\, \mu$, that is, $\mu(y) \geq r$ or $\mu(y) + r > 1$. If $r > 0.5$, then $1 - r < 0.5$, $\mu(y) > 1 - r$, and so $y \in \mu_{1-r}^s$. Since μ_{1-r}^s is an ideal of S by the part (5), we have $xy \in \mu_{1-r}^s$ and $yx \in \mu_{1-r}^s$, that is, $\mu(xy) > 1 - r$ and $\mu(yx) > 1 - r$, so $xy\, q\, [\mu]_r$ and $yx\, q\, [\mu]_r$. If $r \leq 0.5$, then $y \in \mu_r$ and μ_r is an ideal of S by the part (5). Thus, $xy \in \mu_r \subseteq [\mu]_r$ and $yx \in \mu_r \subseteq [\mu]_r$. In any case, we have $xy \in [\mu]_r$ and $yx \in [\mu]_r$. Therefore (5) implies (6).

(6)$\Rightarrow$(1) Assume that (6) holds. For all $x \in S, y_r \in \mu$. If $(xy)_r \overline{\in \vee q}\, \mu$, then $\mu(xy) < r$ and $\mu(xy) + r \leq 1$. Hence $\mu(xy) < 0.5$. Choose t such that $\mu(xy) < t \leq M(r, 0.5)$. Thus $\mu(y) \geq r \geq t$, but $(xy)_t \overline{\in \vee q}\, \mu$, that is, $y \in \mu_t \subseteq [\mu]_t$ but $xy \notin [\mu]_t$, a contradiction. Hence $\forall x \in S, y_r \in \mu \Rightarrow (xy)_r \in \vee q\, \mu$. Similarly, $\forall x \in S, y_r \in \mu \Rightarrow (yx)_r \in \vee q\, \mu$. Therefore (6) implies (1). ∎

Corollary 3.3 Let μ be an $(\in, \in \vee q)$-fuzzy ideal of S. Then $Supp(\mu)$ is an ideal of S.

Naturally, a corresponding result should be considered when μ_r is an ideal of S for all $r \in (0.5, 1]$.

Theorem 3.4 Let μ be any non-empty fuzzy subset in S. Then the following statements are equivalent:

(1) $\mu(xy) \vee 0.5 \geq \mu(x) \vee \mu(x)$ for all $x, y \in S$;

(2) $S \circ \mu \cup \mu \circ S \subseteq \mu \cup 0.5_S$;

(3) $\forall r \in (0.5, 1]$, if μ_r(resp.μ_r^s) is non-empty, then μ_r(resp.μ_r^s) is an ideal of S.

Proof. The proof is analogous to that of Theorem 3.2. ∎

Theorem 3.5 Let A be a non-empty subset in S. Then A is an ideal of S if and only if C_A is an $(\in, \in \vee q)$-fuzzy ideal of S.

Proof. Straightforward. ∎

Theorem 3.6 Let μ and ν be $(\in, \in \vee q)$-fuzzy ideals of S. Then so is $\mu \circ \nu$.

Proof. Assume that μ is an $(\in, \in \vee q)$-fuzzy ideal of S, then by Theorem 3.2, we have $S \circ (\mu \circ \nu) \cap 0.5_S = (S \circ \mu) \circ \nu \cap 0.5_S = (S \circ \mu \cap 0.5_S) \circ \nu \subseteq \mu \circ \nu$. Similarly, assume that ν is an $(\in, \in \vee q)$-fuzzy ideal of S, then $(\mu \circ \nu) \circ S \cap 0.5_S \subseteq \mu \circ \nu$. Therefore, $\mu \circ \nu$ is an $(\in, \in \vee q)$-fuzzy ideal of S. ∎

Theorem 3.7 Let $\{\mu_i | i \in I\}$ be any family of $(\in, \in \vee q)$-fuzzy ideals of S. Then so are $\bigcap_{i \in I} \mu_i$ and $\bigcup_{i \in I} \mu_i$.

Proof. The case for intersection is straightforward. To see that $\bigcup_{i \in I} \mu_i$ is an $(\in, \in \vee q)$-fuzzy ideal of S. Let $\{\mu_i | i \in I\}$ be any family of $(\in, \in \vee q)$-fuzzy ideals of S. Then for all $x \in S$, if $(S \circ \bigcup_{i \in I} \mu_i)(x) = 0$, it is clear that

$$0 = ((S \circ \bigcup_{i \in I} \mu_i) \cap 0.5_S)(x) \leq (\bigcup_{i \in I} \mu_i)(x).$$

Otherwise, there exists $y, z \in S$ such that $x = yz$. Thus, we have

$$\begin{aligned}
((S \circ \bigcup_{i \in I} \mu_i) \cap 0.5_S)(x) &= M((S \circ \bigcup_{i \in I} \mu_i)(x), 0.5) = M(\bigvee_{x=yz} (\bigcup_{i \in I} \mu_i)(z), 0.5) \\
&= M(\bigvee_{x=yz} \bigvee_{i \in I} \mu_i(z), 0.5) = M(\bigvee_{i \in I} \bigvee_{x=yz} \mu_i(z), 0.5) \\
&= \bigvee_{i \in I} \bigvee_{x=yz} M(\mu_i(z), 0.5) \leq \bigvee_{i \in I} \bigvee_{x=yz} \mu_i(yz) \\
&= \bigvee_{i \in I} \mu_i(x) = (\bigcup_{i \in I} \mu_i)(x).
\end{aligned}$$

This implies that $(S\circ\bigcup_{i\in I}\mu_i)\cap 0.5_S \subseteq \bigcup_{i\in I}\mu_i$. Similarly, we have $(\bigcup_{i\in I}\mu_i\circ S)\cap 0.5_S \subseteq \bigcup_{i\in I}\mu_i$. Hence

$$\begin{aligned}((S\circ\bigcup_{i\in I}\mu_i)\cap 0.5_S)\cup((\bigcup_{i\in I}\mu_i\circ S)\cap 0.5_S) =&((S\circ\bigcup_{i\in I}\mu_i)\cup(\bigcup_{i\in I}\mu_i\circ S))\cap 0.5_S\\ \subseteq&\bigcup_{i\in I}\mu_i,\end{aligned}$$

it follows that $\bigcup_{i\in I}\mu_i$ is an $(\in,\in\vee q)$-fuzzy ideal of S. ∎

By the above Theorem, the following Corollary is valid.

Corollary 3.8 The family of all the $(\in,\in\vee q)$-fuzzy ideals of S equipped with fuzzy set inclusion relation " $\subseteq$ " constitutes a complete lattice. And for any $(\in,\in\vee q)$-fuzzy ideals μ and ν of S, $\mu\cap\nu$ and $\mu\cup\nu$ are the greatest lower bound and least upper bound of $\{\mu,\nu\}$, respectively. Its maximal element is C_S. Moreover, it is closed under fuzzy set intersection and union.

Next, we will turn our attention to the relations between the $(\in,\in\vee q)$-fuzzy ideals and the ideals of S. Let us begin with the following Lemma.

Lemma 3.9 Let A be any ideal of S. Then the fuzzy subset μ in S defined by

$$\mu(x)=\begin{cases} r & \text{if } x\in A,\\ t & \text{otherwise.}\end{cases}$$

for all $x\in S$, where $r,t\in[0,1]$ and $r>t$, is an $(\in,\in\vee q)$-fuzzy ideal of S

Proof. Straightforward. ∎

Lemma 3.9 indicates that for any ideal A of S, there must exist $(\in,\in\vee q)$-fuzzy ideals of S whose supporting set are precisely A.

Assume that μ is any $(\in,\in\vee q)$-fuzzy ideal of S. Let $\overline{\mu}=\{\nu|\nu\ \text{is an}\,(\in,\in\vee q)-\text{fuzzy ideal of}\ S\ \ \text{and}\, Supp(\mu)=Supp(\nu)\}$. Then it's easy to see that $\overline{\mu}$ is an equivalence class of the family of all the $(\in,\in\vee q)$-fuzzy ideals of S. Combing this with Corollary 3.3, Theorem 3.6 and Lemma 3.9, we may obtain the following Theorem.

Theorem 3.10 Let $A=\{\overline{\mu}|\mu$ is an $(\in,\in\vee q)$-fuzzy ideal of $S\}$ and $B=\{P|P$ is an ideal of $S\}$. Then the mapping $f:A\to B, f(\overline{\mu})=Supp(\mu)$ defines a isomorphism between A and B under the multiplication defined by $\overline{\mu}\circ\overline{\nu}=\overline{\mu\circ\nu}$.

Proof. By Corollary 3.6 and Lemma 3.15, it is easy to see f is a one-to-one correspondence between A and B. By Theorem 3.9, we know for any $(\in, \in \vee q)$-fuzzy ideals μ and ν, $\mu \circ \nu$ is an $(\in, \in \vee q)$-fuzzy ideal of S, namely $\overline{\mu \circ \nu} \in A$. Hence, it is easy to see the multiplication is well defined. Now, It is not difficult to verify f is a homomorphism. Therefore, the mapping $f : A \to B, f(\overline{\mu}) = Supp(\mu)$ defines a isomorphism between A and B under the multiplication defined by $\overline{\mu} \circ \overline{\nu} = \overline{\mu \circ \nu}$. ∎

4 The $(\in, \in \vee q)$-fuzzy prime ideals of a semigroup

Defition 4.1 Let μ be an $(\in, \in \vee q)$-fuzzy ideal of S. Then μ is said to be

(1) fuzzy prime, if $\forall x, y \in S, r, t \in (0, 1]$, $x_r \circ y_t \in \mu \Rightarrow x_r \in \vee q\ \mu$ or $y_t \in \vee q\ \mu$;

(2) weak-prime, if $\forall x, y \in S, r \in (0, 1]$, $x_r \circ y_r \in \mu \Rightarrow x_r \in \vee q\ \mu$ or $y_r \in \vee q\ \mu$;

(3) semiprime, if $\forall x \in S, r \in (0, 1]$, $x_r^2 \in \mu \Rightarrow x_r \in \vee q\ \mu$.

Theorem 4.2 Let μ be an $(\in, \in \vee q)$-fuzzy prime ideal of S. Then the following statements hold:

(1) $\mu(x) \vee \mu(y) \geq M(\mu(xy), 0.5)\ \ \forall x, y \in S$;

(2) $\forall r \in (0, 0.5]$, if μ_r(resp.μ_r^s) is non-empty, then μ_r(resp.μ_r^s) is a prime ideal of S.

(3) $\forall r \in (0, 1]$, if $[\mu]_r$ is non-empty, then $[\mu]_r$ is a prime ideal of S.

Proof. (1) Assume that μ is an $(\in, \in \vee q)$-fuzzy prime ideal of S. If possible, let $x, y \in S$ such that $\mu(x) \vee \mu(y) < M(\mu(xy), 0.5)$. Choose r such that $\mu(x) \vee \mu(y) < r < M(\mu(xy), 0.5)$. Then $(xy)_r \in \mu$, but $x_r \overline{\in \vee q}\ \mu$ and $y_r \overline{\in \vee q}\ \mu$, a contradiction. Hence the part (1) holds.

(2) Assume that μ is an $(\in, \in \vee q)$-fuzzy prime ideal of S and $r \in (0, 0.5]$ such that μ_r is non-empty. Then μ_r is an ideal of S by the part (5) of Theorem 3.5. For all $x, y \in S$, let $xy \in \mu_r$. Since μ is an $(\in, \in \vee q)$-fuzzy prime ideal of S, then by the part (1) we have $\mu(x) \vee \mu(y) \geq M(\mu(xy), 0.5) \geq M(r, 0.5) = r$. Hence $x \in \mu_r$ or $y \in \mu_r$. Hence μ_r is a prime ideal of S. The case for μ_r^s can be similarly disposed of. Hence the part (2) holds.

(3) Assume that μ is an $(\in, \in \vee q)$-fuzzy prime ideal of S and $r \in (0, 1]$ such that $[\mu]_r$ is non-empty. Then $[\mu]_r$ is an ideal of S by the part (6) of Theorem 3.5. Now, let $xy \in [\mu]_r$ and $x \notin [\mu]_r$. Then $x_r \circ y_r = (xy)_r \in \vee q\, \mu$, $x_r \overline{\in \vee q}\ \mu$, that is, $\mu(x) < r$ and $\mu(x) + r \leq 1$, hence $\mu(x) < 0.5$. If $x_r \circ y_r = (xy)_r \in \mu$, then $y_r \in \vee q\, \mu$ by the Definition of 4.1, that is, $y \in [\mu]_r$. If $x_r \circ y_r = (xy)_r\, q\, \mu$,

then $\mu(xy) > 1 - r$. Since $\mu(x) \vee \mu(y) \geq M(\mu(xy), 0.5)$ by the part (1). We consider the following cases.

Case 1: if $\mu(xy) \geq 0.5$, then $\mu(x) \vee \mu(y) \geq 0.5$. Thus $\mu(y) \geq 0.5$ since $\mu(x) < 0.5$. Now, if $r \leq 0.5$, then $y_r \in \mu$; if $r > 0.5$, then $y_r \, q \, \mu$.

Case 2: if $\mu(xy) < 0.5$, then $\mu(x) \vee \mu(y) \geq \mu(xy) > 1-r$. Thus $\mu(y) > 1-r$ since $\mu(x) + r \leq 1$, and so $y_r \, q \, \mu$.

Therefore, in any case, if $xy \in [\mu]_r$ and $x \notin [\mu]_r$, then $y_r \in \vee q \, \mu$ and so $y \in [\mu]_r$. Hence $[\mu]_r$ is a prime ideal of S and so the part (3) holds. ∎

Theorem 4.3 Let A be a non-empty subset in S. Then A is a prime ideal of S if and only if C_A is an $(\in, \in \vee q)$-fuzzy prime ideal of S.

Remark 4.4 The converse of Theorem 4.2 is not necessarily be true as shown by the following example.

Example 4.5 Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table:

	a	b	c	d
a	a	a	a	a
b	a	b	b	a
c	a	b	c	a
d	a	a	a	d

Let μ be a fuzzy subset in S such that

$$\mu(a) = 0.6, \quad \mu(b) = 0.4, \quad \mu(c) = 0.3, \quad \mu(d) = 0.5.$$

Then

(1) It is easy to check that μ is an $(\in, \in \vee q)$-fuzzy ideal of S and $\mu(x) \vee \mu(y) \geq M(\mu(xy), 0.5)$ for all $x, y \in S$, but μ is not an $(\in, \in \vee q)$-fuzzy prime ideal of S. In fact, $b_{0.5} \circ c_{0.4} = b_{0.4} \in \mu$, but $b_{0.5} \overline{\in \vee q} \, \mu$ and $c_{0.4} \overline{\in \vee q} \, \mu$.

(2) Clearly, $\mu_r = \begin{cases} \{a, d\} & r \in (0.4, 0.5], \\ \{a, b, d\} & r \in (0.3, 0.4], \\ \{a, b, c, d\} & r \in (0, 0.3]. \end{cases}$

$[\mu]_r = \begin{cases} \{a, d\} & r \in (0.4, 0.6], \\ \{a, b, d\} & r \in (0.3, 0.4] or (0.6, 0.7], \\ \{a, b, c, d\} & r \in (0, 0.3] or (0.7, 1]. \end{cases}$

For all $r \in (0, 0.5]$ and $t \in (0, 1]$, it is easy to check that both μ_r and $[\mu]_t$ are fuzzy prime ideals of S, respectively. But μ is not an $(\in, \in \vee q)$-fuzzy prime ideal of S.

But for $(\in, \in \vee q)$-fuzzy weak-prime ideals and semiprime ideals, the next Theorem is valid.

Theorem 4.6 Let μ be an $(\in, \in \vee q)$-fuzzy ideal of S. Then the following statements are equivalent:

(1) μ is an $(\in, \in \vee q)$-fuzzy weak-prime (resp. semiprime) ideal of S;

(2) $\forall x, y \in S$, $\mu(x) \vee \mu(y) \geq M(\mu(xy), 0.5)$(resp.$\mu(x) \geq M(\mu(x^2), 0.5)$);

(3) $\forall r \in (0, 0.5]$, if μ_r(resp.μ_r^s) is non-empty, then μ_r(resp.μ_r^s) is a prime (resp. smeiprime) ideal of S;

(4) $\forall r \in (0, 1]$, if $[\mu]_r$ is non-empty, then $[\mu]_r$ is a prime(resp.smeiprime) ideal of S.

Proof. (1) $\Leftrightarrow$ (2) Assume that μ is an $(\in, \in \vee q)$-fuzzy weak-prime ideal of S. Then by the proof of the part (1) of Theorem 4.2, we know $\forall\, x, y \in S$, $\mu(x) \vee \mu(y) \geq M(\mu(xy), 0.5)$. Hence (1) implies (2). Conversely, assume that the given condition holds. For all $x, y \in S, r \in (0, 1]$, if $x_r \circ y_r = (xy)_r \in \mu$, then $\mu(x) \vee \mu(y) \geq M(\mu(xy), 0.5) \geq M(r, 0.5) = 0.5$ or r according as $r > 0.5$ or $r \leq 0.5$. Hence, either $x_r \in \vee q\ \mu$ or $y_r \in \vee q\ \mu$. Therefore, μ is an $(\in, \in \vee q)$-fuzzy weak-prime ideal of S. Hence (2) implies (1).

(1) $\Leftrightarrow$ (3) Assume that μ is an $(\in, \in \vee q)$-fuzzy weak-prime ideal of S. Then by the proof of the part (2) of Theorem 4.2, we know for all $r \in (0, 0.5]$, non-empty set μ_r is a prime ideal of S. Hence (1) implies (3). Conversely, assume the given condition holds. Then μ is an $(\in, \in \vee q)$-fuzzy ideal of S by Theorem 3.5. Let $x, y \in S$ and $r \leq 0.5$. Now $x_r \circ y_r = (xy)_r \in \mu \Rightarrow xy \in \mu_r \Rightarrow x \in \mu_r$ or $y \in \mu_r$*(since μ_r is an prime ideal of S)*$\Rightarrow x_r \in \mu$ or $y_r \in \mu$. If $r > 0.5$, then since $\mu_{0.5}$ is a prime ideal of S, $x_{0.5} \circ y_{0.5} = (xy)_{0.5} \in \mu \Rightarrow x_{0.5} \in \mu$ or $y_{0.5} \in \mu \Rightarrow x_r\, q\, \mu$ or $y_r\, q\, \mu$. Therefore, μ is an $(\in, \in \vee q)$-fuzzy weak-prime ideal of S. The case for μ_r^s can be similarly disposed of. Hence (3) implies (1).

(1) $\Leftrightarrow$ (4) Assume that μ is an $(\in, \in \vee q)$-fuzzy weak-prime ideal of S. Then by the proof of the part (3) of Theorem 4.2, we know for all $r \in (0, 1]$, non-empty set $[\mu]_r$ is a prime ideal of S. Hence (1) implies (4). Conversely, assume the given condition holds. Let $x, y \in S, t \in (0, 1]$. If $(xy)_r = x_r \circ y_r \in \mu$, then $xy \in \mu_r \subseteq [\mu]_r$. Hence $x \in [\mu]_r$ or $y \in [\mu]_r$, that is $x_r \in \vee q\, \mu$ or $y_r \in \vee q\, \mu$. Therefore, μ is an $(\in, \in \vee q)$-fuzzy weak-prime ideal of S. Hence (4) implies (1).

The case for $(\in, \in \vee q)$-fuzzy semiprime ideal of S can be similarly disposed of. ∎

Corollary 4.7 Let μ be an $(\in, \in \vee q)$-fuzzy weak-prime (resp. semiprime) ideal of S, then $Supp(\mu)$ is a prime (resp. semiprime) ideal of S.

Naturally, a corresponding result should be considered when μ_r is a prime (resp. semiprime) ideal of S for all $r \in (0.5, 1]$.

Theorem 4.8 Let μ be any non-empty fuzzy subset in S. Then the following statements are equivalent:

(1) $\forall x, y \in S$, $\mu(x) \vee \mu(y) \vee 0.5 \geq \mu(xy)$(resp.$\mu(x) \vee 0.5 \geq \mu(x^2)$);

(2) $\forall r \in (0.5, 1]$, μ_r(resp.μ_r^s) is a prime(resp.smeiprime) ideal of S.

Proof. The proof is analogous to that of Theorem 4.6. ∎

Theorem 4.9 Let μ be an $(\in, \in \vee q)$-fuzzy ideal of S. Then the following statements are equivalent:

(1) μ is an $(\in, \in \vee q)$-fuzzy prime ideal of S;

(2) For any fuzzy subsets ν and ω in S, $\nu \circ \omega \subseteq \mu \Rightarrow \nu \subseteq \vee q\, \mu$ or $\omega \subseteq \vee q\, \mu$.

Proof. (1)⇒(2) Assume that μ is an $(\in, \in \vee q)$-fuzzy prime ideal of S. Let ν and ω be fuzzy subsets in S and $\nu \circ \omega \subseteq \mu$. If $\nu \overline{\subseteq \vee q}\, \mu$, then there exists $x_r \in \nu$ such that $x_r \overline{\in \vee q}\, \mu$. Then for all $y_t \in \omega$, $x_r \circ y_t \in \nu \circ \omega \subseteq \mu$, but $x_r \overline{\in \vee q}\, \mu$, hence $y_t \in \vee q\, \mu$*(since μ is an $(\in, \in \vee q)$-fuzzy prime ideal of S).* This implies that $\omega \subseteq \vee q\, \mu$. Hence (1) implies (2).

(2)⇒(1) is clear. ∎

Theorem 4.10 Let μ be an $(\in, \in \vee q)$-fuzzy ideal of S. Then the following statements are equivalent:

(1) μ is an $(\in, \in \vee q)$-fuzzy semiprime ideal of S;

(2) For any fuzzy subset ν in S, $\nu \circ \nu \subseteq \mu \Rightarrow \nu \subseteq \vee q\, \mu$.

Proof. The proof is analogous to that of Theorem 4.9. ∎

Theorem 4.11 Let $\{\mu_i | i \in I\}$ be any family of $(\in, \in \vee q)$-fuzzy weak-prime (resp. semiprime) ideals of S. Then $\bigcup\limits_{i \in I} \mu_i$ is an $(\in, \in \vee q)$-fuzzy weak-prime (resp. semiprime) ideal of S. If $\{\mu_i | i \in I\}$ is any family of $(\in, \in \vee q)$-fuzzy semiprime ideals of S, then $\bigcap\limits_{i \in I} \mu_i$ is an $(\in, \in \vee q)$-fuzzy semiprime ideal of S.

Proof. (1) Let $\{\mu_i | i \in I\}$ be any family of $(\in, \in \vee q)$-fuzzy weak-prime ideals of S. Then $\bigcup\limits_{i \in I} \mu_i$ is an $(\in, \in \vee q)$-fuzzy ideal of S by Theorem 3.10. Now, for all $x, y \in S$

$$\begin{aligned}
(\bigcup_{i \in I} \mu_i)(x) \vee (\bigcup_{i \in I} \mu_i)(y) &= (\bigvee_{i \in I} \mu_i(x)) \vee (\bigvee_{i \in I} \mu_i(y)) = \bigvee_{i \in I} (\mu_i(x) \vee \mu_i(y)) \\
&\geq \bigvee_{i \in I} M(\mu_i(xy), 0.5) = M(\bigvee_{i \in I} \mu_i(xy), 0.5) \\
&= M((\bigcup_{i \in I} \mu_i)(xy), 0.5)
\end{aligned}$$

Hence $\bigcup\limits_{i \in I} \mu_i$ is an $(\in, \in \vee q)$-fuzzy weak-prime ideal of S by Theorem 4.6. The case for $(\in, \in \vee q)$-fuzzy semiprime ideal can be similarly disposed of.

(2) Let $\{\mu_i | i \in I\}$ be any family of $(\in, \in \vee q)$-fuzzy semiprime ideals of S. Then $\bigcap_{i\in I} \mu_i$ is an $(\in, \in \vee q)$-fuzzy ideal of S by Theorem 3.10. Now, for all $x \in S$

$$\begin{aligned}(\bigcap_{i\in I} \mu_i)(x) &= \bigwedge_{i\in I} \mu_i(x) \geq \bigwedge_{i\in I} M(\mu_i(x^2), 0.5) = M(\bigwedge_{i\in I} \mu_i(x^2), 0.5) \\ &= M((\bigcap_{i\in I} \mu_i)(x^2), 0.5)\end{aligned}$$

Hence $\bigcap_{i\in I} \mu_i$ is an $(\in, \in \vee q)$-fuzzy semiprime ideal of S by Theorem 4.6. ∎

In view of Theorem 4.11, we may obtain the following Corollaries.

Corollary 4.12 The family of all the $(\in, \in \vee q)$-fuzzy weak-prime ideals of S equipped with fuzzy set inclusion relation " $\subseteq$ " constitutes a complete lattice. Its maximal element is C_S. Moreover, it is closed under fuzzy set union.

Proof. Let μ and ν be $(\in, \in \vee q)$-fuzzy weak-prime ideals of S. Then by Theorem 4.11, $\mu \cup \nu$ is also an $(\in, \in \vee q)$-fuzzy weak-prime ideal of S and is the least upper bound, while the unique greatest $(\in, \in \vee q)$-fuzzy weak-prime ideal contained in $\mu \cap \nu$, namely the union of the family of all $(\in, \in \vee q)$-fuzzy weak-prime ideals of S contained in $\mu \cap \nu$ is their greatest lower bound. There is no difficulty in replacing the $\{\mu, \nu\}$ with an arbitrary family of $(\in, \in \vee q)$-fuzzy weak-prime ideals of S, and so the family of all the $(\in, \in \vee q)$-fuzzy weak-prime ideals of S equipped with fuzzy set inclusion relation " $\subseteq$ " constitutes a complete lattice. It is clear that its maximal element is C_S and it is closed under fuzzy set union. ∎

Corollary 4.13 The family of all the $(\in, \in \vee q)$-fuzzy semiprime ideals of S equipped with fuzzy set inclusion relation " $\subseteq$ " constitutes a complete lattice. And for any $(\in, \in \vee q)$-fuzzy semiprime ideals μ and ν of S, $\mu \cap \nu$ and $\mu \cup \nu$ are the greatest lower bound and least upper bound of $\{\mu, \nu\}$, respectively. Its maximal element is C_S. Moreover, it is closed under fuzzy set intersection and union.

Lemma 4.14(*Cf.* [15]) Every semiprime ideal of a semigroup can be expressed as the intersection of a family prime ideals of the semigroup.

It is natural to extend this property to $(\in, \in \vee q)$-fuzzy semiprime ideal.

Theorem 4.15 Let μ be an $(\in, \in \vee q)$-fuzzy ideal of S. Then μ is an $(\in, \in \vee q)$-fuzzy semiprime ideal if and only if μ can be expressed as the intersection of a family $(\in, \in \vee q)$-fuzzy prime ideals of S.

Proof. Assume that μ is the intersection of a family $(\in, \in \vee q)$-fuzzy prime ideals of S, then by Theorem 4.11, we know μ is an $(\in, \in \vee q)$-fuzzy semiprime ideal of S *(since the $(\in, \in \vee q)$-fuzzy prime ideal must be an $(\in, \in \vee q)$-fuzzy semiprime ideal).*

Conversely, let μ be an $(\in, \in \vee q)$-fuzzy semiprime ideal of S. Let $p(\mu) = \{\lambda' | \lambda'$ is an $(\in, \in \vee q)$-fuzzy prime ideal of S and $\mu \subseteq \lambda'\}$. Now, let $\lambda = \bigcap_{\lambda' \in p(\mu)} \lambda'$. Obviously, $\mu \subseteq \lambda$. If $\mu \neq \lambda$, then there exists $a \in S$ such that $\mu(a) < \lambda(a)$. Let $\mu(a) < 0.5$. Since μ is an $(\in, \in \vee q)$-fuzzy semiprime ideal of S, then $\mu^s_{\mu(a)}$ is a semiprime ideal of S by Theorem 4.6 and $a \notin \mu^s_{\mu(a)}$, and so there exists a prime ideal P of S such that $\mu^s_{\mu(a)} \subseteq P$ but $a \notin P$ by Lemma 4.14. Now let ν be a fuzzy subset in S such that

$$\nu(x) = \begin{cases} 1 & \text{if } x \in P, \\ \mu(a) & \text{otherwise.} \end{cases}$$

for all $x \in S$. It is easy to check that ν is an $(\in, \in \vee q)$-fuzzy prime ideal of S. Now for all $x \in S$, if $x \in P$, then $\mu(x) \leq 1 = \nu(x)$; if $x \notin P$, then $\mu(x) \leq \mu(a) = \nu(x)$, this implies that $\mu \subseteq \nu$ and so $\nu \in p(\mu)$. Thus $\mu(a) < \nu(a) = \mu(a)$, a contradiction. Next, let $\mu(a) \geq 0.5$ and ω be a fuzzy subset in S such that

$$\omega(x) = \begin{cases} \mu(a) & \text{if } x = a, \\ \mu(x) \vee 0.5 & \text{otherwise.} \end{cases}$$

for all $x \in S$. Then obviously, ω is an $(\in, \in \vee q)$-fuzzy prime ideal of S and $\mu \subseteq \omega$. Thus $\omega \in p(\mu)$ and so $\mu(a) < \omega(a) = \mu(a)$, a contradiction. Therefore, $\mu = \bigcap_{\lambda' \in p(\mu)} \lambda'$. ∎

Obviously, in Lemma 3.15 , if A is a prime(resp.semiprime) ideal of S, then μ is an $(\in, \in \vee q)$-fuzzy weak-prime(resp.semiprime) ideals of S. Combing this with Corollary 4.7, we may obtain the following Theorem.

Theorem 4.16 Let $A = \{\overline{\mu} | \mu$ is an $(\in, \in \vee q)$-fuzzy weak-prime(resp.semiprime) ideal of $S\}$ and $B = \{P | P$ is a prime(resp.semiprime) ideal of $S\}$. Then the mapping $f : A \to B, f(\overline{\mu}) = Supp(\mu)$ defines a one-to-one correspondence between A and B.

Proof. Straightforward. ∎

5 The $(\in, \in \vee q)$-fuzzy maximal ideals of a semigroup

Definition 5.1 An $(\in, \in \vee q)$-fuzzy ideal μ of S is said to be maximal, if for all $(\in, \in \vee q)$-fuzzy ideal λ of S satisfying:

(i) $\mu \subseteq \vee q\, \lambda$,
(ii) $\exists y \in S$ such that $\mu(y) < 0.5$ and $\lambda(y) \geq 0.5$.
then $\lambda(x) \geq 0.5$ for all $x \in S$.

Theorem 5.2 Let μ be an $(\in, \in \vee q)$-fuzzy ideal of S. Then the following statements are equivalent:
(1) μ is an $(\in, \in \vee q)$-fuzzy maximal ideal of S;
(2) $\forall r \in (0, 0.5]$, if μ_r is non-empty, then μ_r is a maximal ideal of S;
(3) $\forall r \in (0, 1]$, if $[\mu]_r$ is non-empty, then $[\mu]_r$ is a maximal ideal of S.

Proof. (1)$\Rightarrow$(2) Assume that (1) holds. If possible, let $r \in (0, 0.5]$ and non-empty set μ_r be not a maximal ideal of S. Then, there exists a ideal A of S such that $\mu_r \subset A \subset S$. Define a fuzzy subset λ in S as follows.

$$\lambda(x) = \begin{cases} 1 & \text{if } x \in A, \\ \mu(x) & \text{otherwise.} \end{cases}$$

for all $x \in S$. Then, it is easy to see that λ is an $(\in, \in \vee q)$-fuzzy ideal of S. Now, we have

(i) Let $x_r \in \mu$. Then $r \leq \mu(x) \leq \lambda(x)$, and so $x_r \in \lambda$. Hence $\mu \subseteq \vee q\, \lambda$.

(ii) Since $\mu_r \subset A \subset S$, then there exists $y \in A/\mu_r$, that is, $\mu(y) < r \leq 0.5$ and $0.5 < \lambda(y) = 1$. On the other hand, if $x \in S/A$, then $\lambda(x) = \mu(x) < r \leq 0.5$, which contradicts the fact that μ is an $(\in, \in \vee q)$-fuzzy maximal ideal of S. Hence (1) implies (2).

(2)$\Rightarrow$(3) Assume that (2) holds. If possible, let $r \in (0, 1]$ and non-empty set $[\mu]_r$ be not a maximal ideal of S. Then there exists an ideal A of S such that $[\mu]_r \subset A \subset S$. If $r \in (0, 0.5]$, then $\mu_r \subseteq [\mu]_r \subset A \subset S$, which contradicts the fact that μ_r is a maximal ideal of S. If $r \in (0.5, 1]$, for all $x \in [\mu]_r$, we have $\mu(x) \geq r > 1 - r$ or $\mu(x) > 1 - r$, thus there exists $\varepsilon > 0$ such that $1 - r + \varepsilon \leq 0.5$ and $\mu(x) \geq 1 - r + \varepsilon$, that is , $x \in \mu_{1-r+\varepsilon}$ and so $[\mu]_r \subseteq \mu_{1-r+\varepsilon}$. On the other hand, for all $x \in \mu_{1-r+\varepsilon}$, we have $\mu(x) \geq 1 - r + \varepsilon > 1 - r$, that is, $x \in [\mu]_r$ and so $\mu_{1-r+\varepsilon} \subseteq [\mu]_r$. Hence $\mu_{1-r+\varepsilon} = [\mu]_r \subseteq [\mu]_r \subset A \subset S$, which contradicts the fact that $\mu_{1-r+\varepsilon}$ is a maximal ideal of S. Hence (2) implies (3).

(3)$\Rightarrow$(1) Assume that (3) holds. If μ is not an $(\in, \in \vee q)$-fuzzy maximal ideal of S. Then there exist an $(\in, \in \vee q)$-fuzzy ideal λ of S satisfying: (i) $\mu \subseteq \vee q\, \lambda$; (ii) $\exists y \in S$ such that $\mu(y) < 0.5$ and $\lambda(y) \geq 0.5$; (iii) $\exists z \in S$ such that $\lambda(z) < 0.5$. Now, choose r such that $\mu(y) \vee \lambda(z) < r < 0.5$. Then $\lambda(y) \geq 0.5 > r > \mu(y)$, that is, $y \in [\lambda]_r/[\mu]_r$ and $z \notin [\lambda]_r$. On the other hand, for all $x \in [\mu]_r$, we have $\mu(x) \geq r$ or $\mu(x) > 1 - r > r$ and so $x_r \in \mu$. Hence $x \in [\lambda]_r$ since $\mu \subseteq \vee q\, \lambda$, and so $[\mu]_r \subseteq [\lambda]_r$. Thus, $[\mu]_r \subset [\lambda]_r \subset S$, a contradiction. Hence (3) implies (1). ∎

Theorem 5.3 Let A be a non-empty set in S. Then A is a maximal ideal of S if and only if C_A is an $(\in, \in \vee q)$-fuzzy maximal ideal of S.

Combing Theorem 4.6 with Theorem 5.2, we may obtain the following Theorem.

Theorem 5.4 Let S be a commutative semigroup such that $S \cdot S = S$. Then every $(\in, \in \vee q)$-fuzzy maximal ideal of S is an $(\in, \in \vee q)$-fuzzy weak-prime ideal of S.

Proof. Assume that μ is an $(\in, \in \vee q)$-fuzzy maximal ideal of S. By Theorem 5.2, for all $r \in (0, 0.5]$, non-empty set μ_r is a maximal ideal of S. Since S is a commutative semigroup and $S \cdot S = S$, hence every maximal ideal of S is a prime ideal, and so for all $r \in (0, 0.5]$, non-empty set μ_r is a prime ideal of S. Thus by Theorem 4.6, we know μ is an $(\in, \in \vee q)$-fuzzy weak-prime ideal of S. ∎

6 Problem of homomorphism

In this section, based on the homomorphism between two crisp semigroups, we will study the properties of the images and inverse images of $(\in, \in \vee q)$-fuzzy ideals, prime(resp.weak-prime, semiprime) ideals and maximal ideals.

Defition 6.1(*Cf.* [2]) Let f be any mapping from a set X into a set X'. A fuzzy subset μ in X is called f-invariant, if for all $x, y \in X$,$f(x) = f(y) \Rightarrow \mu(x) = \mu(y)$.

Clearly, $f^{-1}(f(\mu)) = \mu$, provides that μ is f-invariant.

Theorem 6.2 Let S' be a semigroup, μ and μ' be $(\in, \in \vee q)$-fuzzy ideals of S and S' respectively, and f be a homomorphism from S onto S'. Then

(1) $f(\mu)$ is an $(\in, \in \vee q)$-fuzzy ideal of S';

(2) $f^{-1}(\mu)$ is an $(\in, \in \vee q)$-fuzzy ideal of S;

(3) The mapping $\mu \to f(\mu)$ defines a one-to-one correspondence between the set of the f-invariant $(\in, \in \vee q)$-fuzzy ideals of S and the set of the $(\in, \in \vee q)$-fuzzy ideals of S'.

Proof. (1) Assume that μ is an $(\in, \in \vee q)$-fuzzy ideal of S. For all $x', y' \in S'$, since f is a homomorphism from S onto S', we have

$$\begin{aligned} f(\mu)(x'y') &= \bigvee_{z \in f^{-1}(x'y')} \mu(z) \geq \bigvee_{x \in f^{-1}(x'), y \in f^{-1}(y')} \mu(xy) \\ &\geq \bigvee_{x \in f^{-1}(x'), y \in f^{-1}(y')} M(\mu(x) \vee \mu(y), 0.5) \\ &= M((\bigvee_{x \in f^{-1}(x')} \mu(x)) \vee (\bigvee_{y \in f^{-1}(y')} \mu(y)), 0.5) \\ &= M(f(\mu)(x') \vee f(\mu)(y'), 0.5) \end{aligned}$$

This implies that $f(\mu)$ is an $(\in, \in \vee q)$-fuzzy ideal of S'.

(2) Assume that μ' is an $(\in, \in \vee q)$-fuzzy ideal of S'. For all $x, y \in S$, we have

$$\begin{aligned} f^{-1}(\mu')(xy) &= \mu'(f(xy)) = \mu'(f(x)f(y)) \\ &\geq M(\mu'(f(x)) \vee \mu'(f(y)), 0.5) \\ &= M(f^{-1}(\mu')(x) \vee f^{-1}(\mu')(y), 0.5) \end{aligned}$$

This implies that $f^{-1}(\mu')$ is an $(\in, \in \vee q)$-fuzzy ideal of S.

(3) This part is the direct consequence of the parts (1) and (2). ∎

Theorem 6.3 Let S' be a semigroup, μ and μ' be $(\in, \in \vee q)$-fuzzy prime(resp.weak-prime, semiprime) ideals of S and S' respectively, f be a homomorphism from S onto S'. Then the following statements hold:

(1) $f(\mu)$ is an $(\in, \in \vee q)$-fuzzy prime(resp.weak-prime, semiprime) ideal of S', provided that μ is f-invariant;

(2) $f^{-1}(\mu')$ is an $(\in, \in \vee q)$-fuzzy prime(resp.weak-prime, semiprime) ideal of S;

(3) The mapping $\mu \to f(\mu)$ defines a one-to-one correspondence between the set of the f-invariant $(\in, \in \vee q)$-fuzzy prime(resp.weak-prime, semiprime) ideals of S and the set of the $(\in, \in \vee q)$-fuzzy prime(resp.weak-prime, semiprime) ideals of S'.

Proof. (1) Assume that μ is an $(\in, \in \vee q)$-fuzzy prime ideal of S and f is a homomorphism from S onto S', then $f(\mu)$ is an $(\in, \in \vee q)$-fuzzy ideal of S' by the part (1) of Theorem 6.2. Now, for all $x', y' \in S', r, t \in (0, 1]$, there exists $x, y \in S$ such that $f(x) = x', f(y) = y'$, then $f(x_r) = f(x)_r = x'_r, f(y_t) = f(y)_t = y'_t$. If $x'_r \circ y'_t = (x'y')_{M(r,t)} \in f(\mu)$ but $x'_r \overline{\in \vee q}\ f(\mu)$, then $\mu(xy) = f^{-1}(f(\mu))(xy)$ *(since μ is f-invariant)* $= f(\mu)(f(xy)) = f(\mu)(f(x)f(y)) = f(\mu)(x'y') \geq M(r, t)$ but $\mu(x) = f^{-1}(f(\mu))(x) = f(\mu)(f(x)) = f(\mu)(x') < r$ and $\leq 1 - r$, and so $x_r \circ y_t = (xy)_{M(r,t)} \in \mu$ but $x_r \overline{\in \vee q}\ \mu$. Since μ is an $(\in, \in \vee q)$-fuzzy prime ideal of S, then $y_t \in \vee q\ \mu$ and so $f(\mu)(y') = \bigvee_{a \in f^{-1}(y')} \mu(a) \geq \mu(y) \geq t$ or $> 1 - t$, hence $y'_t \in \vee q\ f(\mu)$. This implies that $f(\mu)$ is an $(\in, \in \vee q)$-fuzzy prime ideal of S'.

(2) Assume that μ' is an $(\in, \in \vee q)$-fuzzy prime ideal of S', then $f^{-1}(\mu')$ is an $(\in, \in \vee q)$-fuzzy ideal of S by the part (2) Theorem 6.2. Now, for all $x, y \in S, r, t \in (0, 1]$, if $x_r \circ y_t = (xy)_{M(r,t)} \in f^{-1}(\mu')$ but $x_r \overline{\in \vee q}\ f^{-1}(\mu')$, then $\mu'(f(xy)) = f(f^{-1}(\mu'))(f(xy)) = \bigvee_{a \in f^{-1}(f(xy))} f^{-1}(\mu')(a) = \bigvee_{a \in f^{-1}(f(xy))} \mu'(f(a)) = \mu'(f(xy)) = f^{-1}(\mu')(xy) \geq M(r, t)$, but
$\mu'(f(x)) = f(f^{-1}(\mu'))(f(x)) = \bigvee_{a \in f^{-1}(f(x))} f^{-1}(\mu')(a) = \bigvee_{a \in f^{-1}(f(x))} \mu'(f(a)) = \mu'(f(x)) = f^{-1}(\mu')(x) < r$ and $\leq 1 - r$, and so $f(x)_r \circ f(y)_t = f(x_r) \circ f(y_t) = f(x_r \circ y_t) = f(xy)_{M(r,t)} \in \mu'$ but $f(x)_r \overline{\in \vee q}\ \mu'$. Since μ' is an $(\in, \in \vee q)$-fuzzy prime ideal of S', then $f(y)_t \in \vee q\ \mu'$ and so $f^{-1}(\mu')(y) = \mu'(f(y)) \geq t$ or

$> 1-t$. Hence $y_t \in \vee q\, f^{-1}(\mu')$. This implies that $f^{-1}(\mu')$ is an $(\in, \in \vee q)$-fuzzy prime ideal of S.

(3) This part is the direct consequence of the parts (1) and (2).

The other cases can be similarly disposed of. ∎

Theorem 6.4 Let S' be a semigroup, μ and μ' be $(\in, \in \vee q)$-fuzzy maximal ideals of S and S' respectively, and f be a homomorphism from S onto S'. If every $(\in, \in \vee q)$-fuzzy ideal of S is f-invariant, then

(1) $f(\mu)$ is an $(\in, \in \vee q)$-fuzzy maximal ideal of S' ;

(2) $f^{-1}(\mu)$ is an $(\in, \in \vee q)$-fuzzy maximal ideal of S;

(3) The mapping $\mu \to f(\mu)$ defines a one-to-one correspondence between the set of the $(\in, \in \vee q)$-fuzzy maximal ideals of S and the set of the $(\in, \in \vee q)$-fuzzy maximal ideals of S'.

Proof. (1) Assume that μ is an $(\in, \in \vee q)$-fuzzy maximal ideal of S, then $f(\mu)$ is an $(\in, \in \vee q)$-fuzzy ideal of S' by the part (1) of Theorem 6.2. If possible, let $f(\mu)$ be not an $(\in, \in \vee q)$-fuzzy maximal ideal of S'. Then there exists an $(\in, \in \vee q)$-fuzzy ideal λ' of S' satisfying: (i) $f(\mu) \subseteq \vee q\, \lambda'$; (ii) $\exists y' \in S'$ such that $f(\mu)(y') < 0.5$ and $\lambda'(y') \geq 0.5$; (iii) $\exists z' \in S'$ such that $\lambda'(z') < 0.5$. By the part (2) of Theorem 6.2, we know $f^{-1}(\lambda')$ is an $(\in, \in \vee q)$-fuzzy ideal of S. Now, we have (i') if $x_r \in \mu$, then $f(x_r) = f(x)_r \in f(\mu) \Rightarrow f(x)_r \in \vee q\, \lambda'$, and so $f^{-1}(\lambda')(x) = \lambda'(f(x)) \geq r$ or $> 1-r$. Hence $x_r \in \vee q\, f^{-1}(\lambda')$ and so $\mu \subseteq \vee q\, f^{-1}(\lambda')$. (ii') Let $y \in S$ such that $f(y) = y'$. Then $\mu(y) = f^{-1}(f(\mu))(y)$ *(since μ is f-invariant by assumption)* $= f(\mu)(f(y)) = f(\mu)(y') < 0.5$ and $f^{-1}(\lambda')(y) = \lambda'(f(y)) = \lambda'(y') \geq 0.5$. (iii') Let $z \in S$ such that $f(z) = z'$. Then $f^{-1}(\lambda')(z) = \lambda'(f(z)) = \lambda'(z') < 0.5$, which contradicts the fact that μ is an $(\in, \in \vee q)$-fuzzy maximal ideal of S. Hence $f(\mu)$ is an $(\in, \in \vee q)$-fuzzy maximal ideal of S'.

(2) Assume that μ' is an $(\in, \in \vee q)$-fuzzy maximal ideal of S', then $f^{-1}(\mu')$ is an $(\in, \in \vee q)$-fuzzy ideal of S by the part (2) of Theorem 6.2. If possible, let $f^{-1}(\mu')$ be not an $(\in, \in \vee q)$-fuzzy maximal ideal of S. Then there exists an $(\in, \in \vee q)$-fuzzy ideal λ of S satisfying: (i) $f^{-1}(\mu') \subseteq \vee q\, \lambda$; (ii) $\exists y \in S$ such that $f^{-1}(\mu')(y) < 0.5$ and $\lambda(y) \geq 0.5$; (iii) $\exists z \in S$ such that $\lambda(z) < 0.5$. By the part (1) of Theorem 6.2, we know $f^{(}\lambda)$ is an $(\in, \in \vee q)$-fuzzy ideal of S'. Now, we have (i') Let $x'_r \in \mu'$ and $x \in S$ such that $f(x) = x'$. Then $f^{-1}(\mu')(x) = \mu'(f(x)) = \mu'(x') \geq r$ and so $x_r \in f^{-1}(\mu')$. Hence $x_r \in \vee q\, \lambda$ and so $f(\lambda)(x') = \bigvee_{a \in f^{-1}(x')} \lambda(a) = \lambda(x)$ *(since λ is f-invariant by assumption)* $\geq r$ or $> 1-r$. Thus $x'_r \in \vee q\, f(\lambda)$ and so $\mu' \subseteq \vee q\, f(\lambda)$. (ii') $\mu'(f(y)) = f^{-1}(\mu')(y) < 0.5$ and $f(\lambda)(f(y)) = \bigvee_{a \in f^{-1}(f(y))} \lambda(a) = \lambda(y) \geq 0.5$. But $f(\lambda)(f(z)) = \bigvee_{a \in f^{-1}(f(z))} \lambda(a) = \lambda(z) < 0.5$, which contradicts the fact that μ' is an $(\in, \in \vee q)$-fuzzy maximal ideal of S'. Hence $f^{-1}(\mu')$ is an $(\in, \in \vee q)$-fuzzy maximal ideal of S.

(3) This part is the direct consequence of the parts (1) and (2). ∎

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References

[1] L.A.Zadeh, Fuzzy sets, Inform and control 8 (1965), 338-353.

[2] A.Rosenfeld, fuzzy groups, J.Math.Anal.Appl.35 (1971), 512-517.

[3] C.V.Negoita and D.A.Ralescu, Applications of Fuzzy Sets to Systems Analysis, Wiwley, New York, 1975.

[4] J.M.Anthony and H.Sherwood, Fuzzy groups redifined, J.Math.Anal.Appl.65 (1979), 124-130.

[5] S.K.Bhakat, P.Das, On the definition of a fuzzy subgroup, Fuzzy sets Syst.51 (1992), 235-241.

[6] S.K.Bhakat, P.Das, $(\in, \in \vee q)$-Fuzzy subgroup, Fuzzy sets Syst.80 (1996), 359-368.

[7] S.K.Bhakat, P.Das, Fuzzy subrings and ideals redefined, Fuzzy sets Syst.81 (1996), 383-393.

[8] S.K.Bhakat, $(\in, \in \vee q)$-Fuzzy normal, quasinormal and maximal subgroups, Fuzzy sets Syst.112 (2000), 299-312.

[9] Y.B. Jun, S.Z. Song, Generalized fuzzy interior ideals in semigroups, Inform. Sci.176 (2006), 3079-3093.

[10] Kuroki N. On fuzzy semigroups. Inform. Sci.53 (1991), 203-236.

[11] Kuroki N. On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy sets Syst.5 (1981), 203-205.

[12] Kuroki N. On fuzzy semiprime ideals in semigroups, Fuzzy sets Syst.8 (1982), 71-79.

[13] Mo Z W, Wang X P, Fuzzy ideals generated by fuzzy sets in semigroups, Inform. Sci.86 (1995), 203-210.

[14] P.P.Ming,L.Y.Ming, Fuzzy topology 1. Neighbourhood strctures of a fuzzy point and Moore-Smith convergence,J.Math.Anal.Appl.76 (1980), 571-579.

[15] Guo Y G, Shum K P. Completely semiprime and completely prime congruences on semigroups, Words, Langruages and Combinaorics. River Edge, N.J.:Word Science Publishing,1992, 179-186.

[16] Xie X-Y. On fuzzy ideals of a semigroup, J.Fuzzy Math.,1999,7 (2), 357-

365.

[17] Xie X-Y. On prime fuzzy ideals of a semigroup, J.Fuzzy Math.,2000,8 (4), 231-241.

[18] Xie X-Y. Fuzzy ideals in semigroups, J.Fuzzy Math.,(7) 1999, 357-365.

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