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Coincidence and Common Fixed Points in Symmetric Spaces under Implicit Relation and Application

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Abstract

In this paper, we prove some coincidence and common fixed point results by using an implicit relation for weakly compatible mappings in symmetric space. The result of Imdad et. al. [6] is improved. Our theorem generalizes some related results of symmetric space. We give an application to our Corollary into the probabilistic metric spaces.

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1 Introduction

Hicks and Rhoades [5] established some common fixed point theorems in symmetric space by using the fact that *some of the properties of metric are not required in the proofs of certain metric theorems*. Taking this view into account, Aliouche [1], Aamri and Moutawakil [4] and Imdad et al. [6] etc. proved some common fixed point results under weakly compatible maps introduced

by Jungck [8]. This was done by using an independent notion than weakly compatible maps, called property (EA) by Aamri and Moutawakil [3].

In this paper, we prove some coincidence and common fixed point results by using an implicit relation for weakly compatible mappings in symmetric space. An application is given for symmetric PPM-space.

2 Preliminaries

Definition 2.1 *A symmetric on a set X is a function $d : X \times X \to [0, \infty)$ such that for all $x, y \in X$*
(i) $d(x, y) = 0 \iff x = y$,
(ii) $d(x, y) = d(y, x)$.

Definition 2.2 *[10] Let (X, d) be a symmetric space then the symmetric d satisfies the following conditions:*
(W.3) Given $\{x_n\}, x, y$ in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0 \Rightarrow x = y$;
(W.4) Given $\{x_n\}, \{y_n\}$ and an x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0 \Rightarrow \lim_{n\to\infty} d(y_n, x) = 0$.

Now we state some commuting type conditions from the literature which are relevant in the present context and can be adopted from metric space.

Definition 2.3 *[7] A pair of self-maps (f, g) on a symmetric space (X, d) is said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.*

Obviously (f, g) is noncompatible ([9]) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = t$ and $\lim_{n\to\infty} gx_n = t$, for some $t \in X$ but $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is either nonzero or nonexistent. Noncompatible maps is further generalized to property (EA), while compatible maps is generalized to weakly compatible.

Definition 2.4 *[8] A pair of self-maps (f, g) of a symmetric space (X, d) is said to weakly compatible (or, co-incidently commuting) if they commute at their coincidence point, i.e., $fgx = gfx$ whenever $fx = gx$.*

Definition 2.5 *[4] A pair of self-maps (f, g) on a symmetric space (X, d) is said to enjoy property (E.A) if there exist a sequence $\{x_n\}$ such that $\lim_{n\to\infty} d(fx_n, t) = \lim_{n\to\infty} d(gx_n, t) = 0$, for some $t \in X$.*

Property (E.A) need not imply Noncompatibility. To show this, let $X = [1, \infty)$ and d be a symmetric on X defined by $d(x, y) = (x-y)^2$. Define $A, S : X \to X$ by: $Ax = 2x - 1$ and $Sx = x^2$, for all $x, y \in X$. Then for a sequence $\{x_n\} =$

$\{1+\frac{1}{n}\}$ we have $\lim_{n\to\infty} d(Ax_n, 1) = 0$ and $\lim_{n\to\infty} d(Sx_n, 1) = 0$. So (A, S) satisfies property (E.A). Also $\lim_{n\to\infty} d(ASx_n, SAx_n) = \lim_{n\to\infty} d(2x_n^2 - 1, (2x_n - 1)^2) = \lim_{n\to\infty} 4(x_n^2 - 1)^4 = 0$. Hence (A, S) is compatible. It follows the assertion.

Definition 2.6 *[1] Let (X, d) be a symmetric space. We say that (X, d) satisfies property (H.E) if given $\{x_n\}, \{y_n\}$ and x in X,*
$\lim_{n\to\infty} d(x_n, x) = 0$ *and* $\lim_{n\to\infty} d(y_n, x) = 0 \Rightarrow \lim_{n\to\infty} d(x_n, y_n) = 0$.

Suppose $X = [0, +\infty)$ and the symmetric function d is defined by:
$d(x, y) = e^{|x-y|} - 1, \quad \forall x, y \in X$. Then (X, d) satisfies property (H.E).

3 Implicit Relations

Let $\mathcal{F}_5$ is the set of all non-increasing, continuous functions $F(t_1, t_2, t_3, t_4, t_5) : \mathbf{R}_+^5 \mapsto \mathbf{R}$ satisfying the following conditions:
(C_1): For all $u \geq 0$ and $v > 0$ we have either
$(C_{1a}) : \quad F(u, v, u, v, 0) \leq 0 \Rightarrow u < v$,
or $(C_{1b}) : \quad F(u, v, v, u, 0) \leq 0 \Rightarrow u < v$.
(C_2): For all $u > 0$ we have
$(C_{2a}) : \quad F(u, 0, u, 0, 0) > 0$,
and $(C_{2b}) : \quad F(u, 0, 0, u, 0) > 0$.
$(C_3) : \quad F(u, u, 0, 0, u) \leq 0 \Rightarrow u = 0$.

Let us denote the implicit relation satisfying $(C_1) - (C_3)$ above, by (I). Following are some examples of implicit relations:

Example 3.1 *Let $F(t_1, ..., t_5) = pt_1 - qt_2 - r(t_3 + t_4) - st_5$, where $p, q, r, s \geq 0$, $p > q + s > r$ and $0 \leq q + 2r < p$.*

For (C_1) : if $u \geq 0$, $v > 0$ then:
$(C_{1a}) : F(u, v, u, v, 0) = pu - qv - r(u + v) \leq 0 \Rightarrow u \leq (\frac{q+r}{p-r})v < v$.
(C_{1b}) : like (C_{1a}).
$(C_{2a}) : F(u, 0, u, 0, 0) = u(p - r) > 0$ whenever $u > 0$, as $p > r$.
(C_{2b}) : like (C_{2a}).
$(C_3) : F(u, u, 0, 0, u) \leq 0 \Rightarrow (p - q - s)u \leq 0 \Rightarrow u = 0$, as $q + s < p$.

Example 3.2 *Let $F(t_1, ..., t_5) = t_1 - \phi(max\{t_2, \frac{k}{2}(t_3 + t_4), t_5\})$ where $1 \leq k < 2$ and let $\phi : \mathbf{R}_+ \mapsto \mathbf{R}_+$ be a non-decreasing function such that $\phi(t) < t$ for each $t \in (0, +\infty)$. Then F satisfies all the conditions $(C_1) - (C_3)$ of (I).*

Taking $t_1 = d(Ax, By)$, $t_2 = d(Sx, Ty)$, $t_3 = d(Ax, Sx)$, $t_4 = d(By, Ty)$ and $t_5 = min\{d(By, Sx), d(Ax, Ty)\}$; it is easy to check the conditions $(C_1) - (C_3)$. For, suppose that $u \geq 0$ and $v > 0$, then for (C_{1a}) :

$F(u,v,u,v,0) = u-\phi(max\{v, \frac{k}{2}(u+v), 0\}) \leq 0 \Rightarrow u \leq \phi(max\{v, \frac{k}{2}(u+v), 0\})$.

By taking $\frac{k}{2}(u+v)$ as 'max' we obtain $u \leq \phi(\frac{k}{2}(u+v)) < \frac{k}{2}(u+v) \Rightarrow u < (\frac{k}{2-k})v \not< v$, a contradiction; since $1 \leq (\frac{k}{2-k}) < \infty, \ \forall\ k \in [1,2)$. Similarly 0 can't be 'max', as $v > 0$. Lastly, if v is the 'max' then it implies that $u \leq \phi(v) < v$, which is true; as $max\{v, \frac{k}{2}(u+v), 0\} = v \Rightarrow v > \frac{k}{2}(u+v) \Rightarrow u < (\frac{2-k}{k})v \leq v, \ \forall k \in [1,2)$. Thus (C_{1a}) is satisfied, for all $u \geq 0$ and $v > 0$.

(C_{1b}): like(C_{1a}).

(C_{2a}) : We have $F(u,0,u,0,0) = u - \phi(max\{0, ku/2, 0\}) = u - \phi(ku/2) > u - ku/2 > 0$, for all $u > 0$, as $1 \leq k < 2$. Thus (C_{2a}) is satisfied.

(C_{2b}) : like (C_{2a}).

$(C_3) : F(u,u,0,0,u) = u - \phi(max\{u,0,u\} = u - \phi(u) \leq 0$. If $u > 0$ then this gives $u \leq \phi(u) < u$, a contradiction. Thus $u = 0$. Hence (C_3) is satisfied.

Example 3.3 *Let $F(t_1, ..., t_5) = t_1 - \phi(max\{t_2, \frac{k}{2}(t_3+t_4), t_5\})$ where $F \in \mathcal{F}_5$. If $k \in [1,2)$ and $\phi : \mathbf{R}_+ \mapsto \mathbf{R}_+$ is a non-decreasing function such that $\phi(t) < t$ for each $t \in (0, +\infty)$; then the inequality:*

$$d(Ax, By) \leq \phi(max\{d(Sx,Ty), \tfrac{k}{2}(d(Ax,Sx) + d(By,Ty)), \tfrac{k}{2}[d(By,Sx) + d(Ax,Ty)]\})$$

satisfies all the conditions $(C_1) - (C_3)$ of the implicit relation (I).

Putting $t_1 = d(Ax,By)$, $t_2 = d(Sx,Ty)$, $t_3 = d(Ax,Sx)$, $t_4 = d(By,Ty)$ and $t_5 = \frac{k}{2}[d(By,Sx) + d(Ax,Ty)]$. It is easy to check the conditions $(C_1) - (C_3)$, as $t_5 = 0$ for each (C_{1a}), (C_{1b}), (C_{2a}) and (C_{2b}).

Now, $(C_3) : F(u,u,0,0,u) = u - \phi(max\{u,0,ku/2\} = u - \phi(u) \leq 0$.

If $u > 0$ then it gives $u \leq \phi(u) < u$, a contradiction. Thus $u = 0$. So (C_3) is also satisfied. Thus the inequality satisfies $(C_1) - (C_3)$ of the implicit relation.

Example 3.4 *Taking $\frac{d(By,Sx)+d(Ax,Ty)}{2}$ instead of $\frac{k}{2}[d(By,Sx) + d(Ax,Ty)]$ for t_5 in above Example 3.3, is also true for (I), in the similar way.*

Remark 3.5 *The argument $d(Ay_m, Bx_m) < d(Ay_m, Sy_m)$ to obtain $Ay_m \to t$, in Theorem 2.4 of Imdad et. al. [6] on page 358, is not valid because of the presence of term $\frac{k}{2}[d(Ax,Ty) + d(By,Sx)]$, in the inequality (iii). This term may be replaced by the term $min\{d(Ax,Ty), d(By,Sx)\}$. It is done in our Theorems below, and as shown in Example 4.4, the improved inequality (4.4.1) holds the implicit relation (I).*

4 Main Results

Theorem 4.1 *Let A, B, S and T be self-mappings of a symmetric space (X,d) that enjoys (W.3), (W.4) and (H.E). Suppose that*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$.
(ii) The inequality
$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), (By, Ty), min\{d(By, Sx), d(Ax, Ty)\}) \leq 0$,
holds, for all $x, y \in X$, *where* $F \in \mathcal{F}_5$ *satisfy the implicit relation (I) .*
(iii) The pair (B, T) *enjoys property (E.A) and pair* (A, S) *is weakly compatible (or, alternatively the pair* (A, S) *enjoys property (E.A) and pair* (B, T) *is weakly compatible).*

If (iv): $S(X)$ *is a d-closed subset of* X *(or, alternatively* $T(X)$ *is a d-closed subset of* X*) then the pairs* (A, S) *and* (B, T) *have a coincidence point.*

Proof. Since the pair (B, T) enjoys property (E.A), therefore there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(Bx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0$, for some $t \in X$. Using property (H.E) this yields $\lim_{n\to\infty} d(Bx_n, Tx_n) = 0$. Further, since $B(X) \subseteq S(X)$, for this $\{x_n\}$ there exists some $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence $\lim_{n\to\infty} d(Sy_n, t) = 0$.

Let us show that $\lim_{n\to\infty} d(Ay_n, t) = 0$. If not, then there may exist a sequence $\{Ay_m\}$ of $\{Ay_n\}$, a positive number M and a number $\gamma > 0$ such that for each $m \geq M$, we have $d(Ay_m, t) \geq \gamma$ and $d(Ay_m, Bx_m) \geq \gamma$. So, from (ii), we have

$$F(d(Ay_m, Bx_m), d(Sy_m, Tx_m), d(Ay_m, Sy_m), d(Bx_m, Tx_m), \\ min\{d(Bx_m, Sy_m), d(Ay_m, Tx_m)\}) \leq 0,$$

i.e., $F(d(Ay_m, Bx_m), d(Bx_m, Tx_m), d(Ay_m, Bx_m), d(Bx_m, Tx_m), 0) \leq 0$.
Using (C_{1a}) this relation implies that

$$d(Ay_m, Bx_m) < d(Bx_m, Tx_m).$$

Now for large m, the right hand side of it can be made as small as we wish, so that we have $d(Bx_m, Tx_m) < \gamma$. Thus we have

$$\gamma \leq d(Ay_m, Bx_m) < d(Bx_m, Tx_m) < \gamma,$$

a contradiction. Thus $\lim_{n\to\infty} d(Ay_n, t) = 0$. Hence we obtain,

$$\lim_{n\to\infty} d(Ay_n, t) = 0 = \lim_{n\to\infty} d(Bx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = \lim_{n\to\infty} d(Sy_n, t) \qquad (4.1.1)$$

Now suppose that $S(X)$ is a $d-$closed subset of X so that $Sy_n \to t$ where $t \in S(X)$, and one can find a point $u \in X$ such that $t = Su$. We claim that $Au = Su$. If not, then from inequality (ii), we have

$$F(d(Au, Bx_n), d(Su, Tx_n), d(Au, Su), d(Bx_n, Tx_n), \\ min\{d(Bx_n, Su), d(Au, Tx_n)\}) \leq 0.$$

Since F is a non-increasing and continuous function, making $n \to \infty$ this yields

$$F(d(Au, Su), 0, d(Au, Su), 0, min\{0, d(Au, Su)\}) \leq 0,$$

or, $$F(d(Au, Su), 0, d(Au, Su), 0, 0\}) \leq 0,$$

a contradiction of (C_{2a}). Thus $Au = Su$. Next, since the pair (A, S) is weakly compatible, $Au = Su \Rightarrow ASu = SAu$. Also from (i), $A(X) \subseteq T(X)$, so there exist a point $w \in X$ such that $Au = Tw$. We assert that $Bw = Tw$. If not, then using inequality (ii), we have

$$F(d(Au, Bw), d(Su, Tw), d(Au, Su), d(Bw, Tw), \\ min\{d(Bw, Su), d(Au, Tw)) \leq 0,$$

or, $\quad F(d(Au, Bw), 0, 0, d(Au, Bw), 0) \leq 0,$

a contradiction of (C_{2b}). Thus $Au = Su = Bw = Tw$; showing that the pairs (A, S) and (B, T) have coincidence points u and w, respectively. The proof is similar if we consider the alternative case. For this, we use the alternative condition (C_{1b}) to obtain the coincidence point. This completes the proof.

Now, we give the following fixed point theorem in the setting of Theorem 4.1:

Theorem 4.2 *In the setting of Theorem 4.1, the mappings A, B, S and T have a unique common fixed point, provided one adds the weak compatibility of the pair (B, T) (or alternatively, weak compatibility of the pair (A, S)).*

Proof. In view of Theorem 4.1, we conclude that

$$Au = Su = Bw = Tw = t. \tag{4.2.1}$$

Since (A, S) is weakly compatible, we have $At = St$. We obtain $At = St = t$; by putting $x = t$, $y = w$ in (ii) and using (C_3). Thus t is a common fixed point of A and S. Now, the weak compatibility of (B, T) implies that $BTw = TBw$, i.e., $Bt = Tt$. By putting $x = u$ and $y = t$ in inequality (ii), and using (C_3) we obtain $d(Bt, t) = 0$; whence $Bt = Tt = t$. Thus t is a common fixed point of A, B, S and T. The uniqueness of t follows easily by (C_3). The proof is similar for the other case. This completes the proof.

Next, we slightly change the condition of d-closeness of d to obtain the following result. We start with some definitions (see [2]), regarding this theorem:

Let (X, d) be a symmetric space. A sequence $\{x_n\}$ in X is called a *d-Cauchy sequence*, if it satisfies the usual metric condition. X is called *S-complete* if for every d-Cauchy sequence $\{x_n\}$, there exists an x in X such that $\lim\limits_{n\to\infty} d(x_n, x) = 0$.

Let (X, d) be a symmetric space and let $\{x_n\}$ be a d-Cauchy sequence. Note that, if X is S-complete, then there exists $x \in X$ such that $\lim\limits_{n\to\infty} d(x_n, x) = 0$. Therefore S-completeness of (X, d) implies d-Cauchy completeness of X ([2]).

Theorem 4.3 *Let (X, d) be a symmetric space which satisfies (W.3), (W.4) and (H.E). Let A, B, S and T be self-maps of X satisfying (i) and (ii). Suppose (A, S) or (B, T) satisfy property (E.A), and (A, S) and (B, T) are weakly compatible.*

If (v): the range of one of the mappings A, B, S and T is S-complete subspace of X, then A, B, S and T have a unique common fixed point in X.

Proof. Suppose that $S(X)$ is an S-complete subspace of X. Then $t = Su$, for some $u \in X$. Consequently, we obtain equation (4.1.1). Thus t will be a unique common fixed point of A, B, S and T. That is;

$$At = Bt = St = Tt = t. \tag{4.3.1}$$

The proof is similar when $T(X)$ is assumed to be an S-complete subspace of X. The case in which $A(X)$ or $B(X)$ is S-complete subspace of X is similar to the case in which $T(X)$ or $S(X)$ respectively, is complete, as $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. This completes the proof.

Following example illustrates our Theorem 4.2 (Theorem 4.3):

Example 4.4 *Let F be as in Example 3.2, and satisfy the inequality*

$$d(Ax, By) \leq \phi(m(x, y)) \ \forall \ x, y \in X, \tag{4.4.1}$$

where $m(x, y) =$

$$max\{d(Sx, Ty), \tfrac{k}{2}(d(Ax, Sx)+d(By, Ty)), min\{d(By, Sx), d(Ax, Ty)\}\}.$$

If the remaining conditions of Theorem 4.2 (Theorem 4.3) holds, then the mappings A, B, S and T have a unique common fixed point in X.

As seen in Example 3.2, $F \in \mathcal{F}_5$ satisfies the conditions $(C_1) - (C_3)$. So the inequality (4.4.1) satisfies the implicit relation (I). Now, the inequality (4.4.1) with remaining conditions of Theorem 4.2 (Theorem 4.3), provides the unique common fixed point of A, B, S and T in X.

Remark 4.5 *The contractive conditions (2.4.1) and (2.5.1) of Theorems 2.4 and 2.5, respectively of Imdad et. al. [6] are replaced in our Theorems. Further, as shown in Example 4.4 that, the improved inequality satisfies the implicit relation (I), so we have improved and generalized the Theorems 2.4 and 2.5 of Imdad et. al. [6]. However, the contractive conditions (2.4.1) and (2.5.1) of [6] satisfies the implicit relation (I), as shown in Examples 3.3 and 3.4 respectively.*

Following is also an illustrative example to our Theorems 4.2 and 4.3:

Example 4.6 *Let $F(t_1, ..., t_5) = t_1 - \psi(max\{t_2, t_3, t_4, t_5\})$ for all $t_1, ..., t_5 \geq 0$ and $F \in \mathcal{F}_5$. Further, let $\psi : \mathbf{R}_+ \mapsto \mathbf{R}_+$ be a non-decreasing and right continuous function such that $\lim\limits_{n\to\infty} \psi^n(t) = \infty$. Suppose $t_1 = d(Ax, By)$, $t_2 = d(Sx, Ty)$, $t_3 = d(Ax, Sx)$, $t_4 = d(By, Ty)$ and $t_5 = min\{d(By, Sx), d(Ax, Ty)\}$.*

It is easy to see that under these conditions, the function ψ satisfies also $\psi(t) < t$ for each $t \in (0, +\infty)$. Thus the inequality

$$d(Ax, By) \leq \psi(w(x, y)) \ \ \forall x, y \in X, \tag{$*$}$$

where

$w(x, y) = max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)), \ min\{d(By, Sx), d(Ax, Ty)\}),$

satisfies all the conditions $(C_1)-(C_3)$ of the implicit relation (I). If the remaining conditions are same as in Theorem 4.2 (Theorem 4.3), then the mappings A, B, S and T have a unique common fixed point in X.

Putting $A = B$, $S = T$ in Example 4.6, it reduces to the following Corollary:

Corollary 4.7 *Let (X, d) be a symmetric space that satisfies (W.3), (W.4) and property (H.E). Let A and S be two weakly compatible self maps of X, that satisfies property (E.A) and the following conditions:*
(i) $A(X) \subseteq S(X)$,
(ii) $d(Ax, Ay) \leq \psi(max\{d(Sx, Sy), d(Ax, Sx), d(Ay, Sy),$
$min\{d(Ay, Sx), d(Ax, Sy)\}\})$
for all $x, y \in X$, and $F \in \mathcal{F}_5$ with conditions $(C_1) - (C_3)$;
where $\psi : \mathbf{R}_+ \mapsto \mathbf{R}_+$ is such that $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$.

If $A(X)$ or $S(X)$ is an S-complete subspace of X, then A and S have a unique common fixed point in X.

We will use this result to prove an analogues common fixed point result for a symmetric PPM-space in the following section.

5 Application

A distribution function f is a non-decreasing, left-continuous, real-valued function defined on the set of real numbers, with $\inf f = 0$ and $\sup f = 1$. The distribution function H is defined by $H(t) = 0$ if $t \leq 0$ and $H(t) = 1$ if $t > 0$.

Definition 5.1 *[2] Let X be a set and $\Im$ a function defined on $X \times X$ such that $\Im = F_{x,y}$ is a distribution function. Consider the following conditions:*
I. *$F_{x,y}(0) = 0$ for all $x, y \in X$.*
II. *$F_{x,y} = H$ if and only if $x = y$.*
III. *$F_{x,y} = F_{y,x}$.*
IV. *$F_{x,y}(\epsilon) = 1$ and $F_{y,z}(\delta) = 1$ then $F_{x,z}(\epsilon + \delta) = 1$.*

If $\Im$ satisfies **I** and **II**, then $(X, \Im)$ is called a *PPM-space*, while $\Im$ satisfying **III** is said to be symmetric. A symmetric PPM-space $(X, \Im)$ satisfying **IV** is called a probabilistic metric space. Throughout this section, let $(X, \Im)$ is a symmetric PPM-space. Recall that a sequence $\{x_n\}$ is called a *fundamental sequence* if $\lim_{n\to\infty} F_{x_m,x_n}(t) = 1$ for all $t > 0$ $m > n$ and $m, n \in \mathbf{N}$. The space $(X, \Im)$ is called *F-complete* if for every fundamental sequence $\{x_n\}$, there exists an $x \in X$ such that $\lim_{n\to\infty} F_{x_n,\, x}(t) = 1$ for all $t > 0$. The conditions of Wilson [10] and properties (H.E) and (E.A) are equivalent, in symmetric PPM-space, to the following:
$(W3)$: $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ and $\lim_{n\to\infty} F_{x_n,y}(t) = 1$ imply $x = y$.

$(W4)$: Given $\{x_n\}, \{y_n\}$ and x in X, $\lim\limits_{n\to\infty} F_{x_n,x}(t) = 1$, $\lim\limits_{n\to\infty} F_{x_n,y_n}(t) = 1 \Rightarrow \lim_{n\to\infty} F_{y_n,x}(t) = 1,\ \forall t > 0$.
(HE): Given $\{x_n\}, \{y_n\}$ and x in X, $\lim_{n\to\infty} F_{x_n,x}(t) = 1$, $\lim_{n\to\infty} F_{y_n,x}(t) = 1 \Rightarrow \lim_{n\to\infty} F_{x_n,y_n}(t) = 1,\ \forall t > 0$.
(EA): The pair (A, S) of self-maps of a symmetric PPM-space $(X, \Im)$ is said to satisfy property (EA) if there exist a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} F_{Ax_n,p}(t) = 1$ and $\lim\limits_{n\to\infty} F_{Sx_n,p}(t) = 1,\ \forall t > 0$ and for some $p \in X$.

Corollary 5.2 *[2] Hicks and Rhoades [5] proved that each symmetric PPM-space $(X, \Im)$ admits a compatible symmetric function d as follows:*
$d(x, y) = 0$ *if* $y \in N_x(t, t)$ *for all* $t > 0$, *and*
$d(x, y) = sup\{t : y \notin N_x(t, t),\ 0 < t < 1\}$ *otherwise.*

Then (1) $d(x, y) < t$ if and only if $F_{x,y}(t) > 1 - t$,
(2) d is a compatible symmetric for $t(\Im)$, and
(3) $(X, \Im)$ is F-complete if and only if (X, d) is S-complete.

Now we are in position to prove the following result of this section:

Theorem 5.3 *Let $(X, \Im)$ be a symmetric PPM-space that satisfies $(W3)$, $(W4)$ and (HE). Let A and S be two weakly compatible self-maps of X that satisfies property (EA), such that:*
(I) $A(X) \subseteq S(X)$,
(II) $min\{F_{Sx,Sy}(t), F_{Ax,Sx}(t), F_{Ay,Sy}(t), max(F_{Ay,Sx}(t), F_{Ax,Sy}(t))\} > 1 - t$ implies $F_{Ax,Ay}(\psi(t)) > 1 - \psi(t)$ $\forall t > 0$, and for all $x, y \in X$.

If one of $A(X)$ or $S(X)$ is an F-complete subspace of X, then A and S have a unique common fixed point in X.

Proof. Let us show that, all the conditions of this Theorem reduces to the respective conditions of Corollary 4.7. We first show that inequality (II) reduces to inequality (ii). Recall that the condition (ii) of Corollary 4.7 satisfies the implicit relation (I), as shown in Example 4.6. By using the view of compatible symmetric we will show that inequality (II) is equivalent to the inequality (ii). For this, let $\epsilon > 0$ be given, and sat that:
$t = \epsilon + max\{d(Sx, Sy), d(Ax, Sx), d(Ay, Sy),\ min\{d(Ay, Sx), d(Ax, Sy)\}\}$. $(**)$
Then $(**)$ gives that

$$max\{d(Sx, Sy), d(Ax, Sx), d(Ay, Sy),\ min\{d(Ay, Sx), d(Ax, Sy)\}\} < t,$$

whence; each of the factors

$d(Sx, Sy), d(Ax, Sx), d(Ay, Sy)$ and $min\{d(Ay, Sx), d(Ax, Sy)\}$ are $< t$.

This, by compatible symmetric, implies that
each factors $F_{Sx,Sy}(t), F_{Ax,Sx}(t), F_{Ay,Sy}(t)$ and $max(F_{Ay,Sx}(t), F_{Ax,Sy}(t)) > 1-t$,
or jointly,

$$min\{F_{Sx,Sy}(t), F_{Ax,Sx}(t), F_{Ay,Sy}(t),\ max(F_{Ay,Sx}(t), F_{Ax,Sy}(t))\} > 1 - t.$$

By using inequality (II) it implies that

$$F_{Ax,Ay}(\psi(t)) > 1 - \psi(t), \quad \text{for all } x, y \in X \text{ and for all } t > 0.$$

Using (1) and then $(**)$, this implies that

$$d(Ax, Ay) < \psi(t) = \psi(\epsilon + max\{d(Sx, Sy), d(Ax, Sx), d(Ay, Sy), min\{d(Ay, Sx), d(Ax, Sy)\}\}).$$

Since ψ is right continuous function, so letting $n \to \infty$ this yields

$$d(Ax, Ay) \leq \psi(max\{d(Sx, Sy), d(Ax, Sx), d(Ay, Sy), min\{d(Ay, Sx), d(Ax, Sy)\}\}),$$

for all $x, y \in X$. This is the inequality (ii) of Corollary 4.7, as required.

Further, suppose that $S(X)$ is an S-complete subspace of X then, by definition, $S(X)$ is d-Cauchy complete. Let $\{x_n\} \in X$, so that $d(Sx_n, u) \to 0$ for some $u \in X$. Since (A, S) satisfy property (EA) so that $\lim_{n\to\infty} d(Sx_n, u) = 0 = \lim_{n\to\infty} d(Ax_n, u)$. Now, $A(X) \subseteq S(X)$, one can find some $v \in X$ such that $Au = Sv$. The claim $Au = Av$ follows easily, by using inequality (ii). Uniqueness of v such that $Av = Sv = v$, follows by (ii). Similar argument applies for the F-completeness of $A(X)$, as it is d-Cauchy complete. The proof is complete by Applying Corollary 4.7.

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