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An Error Bound on Uniform Approximation of Bounded Function by Bernstein Polynomial

Vasana Sukkrasanti and Praparnrat Lerdkasem

Department of Mathematics, Faculty of Science
Chulalongkorn University, Bangkok 10330, Thailand
Vasana.C@chula.ac.th

Abstract

Let $f : [0,1]^p \to \mathbb{R}^q$ be a bounded function. In this paper, we used technique from [11] to give a bound on uniform approximation of bounded function f by Bernstein polynomial. The bound is of the form $C\omega(\frac{1}{\sqrt{n}})$ where C is a constant and $\omega(\frac{1}{\sqrt{n}})$ is a modulus of f depend on $\frac{1}{\sqrt{n}}$.

Keywords: Bernstein polynomial, uniform approximation, Weierstrass approximation, modulus of function

1 Introduction

In this paper, we investigated a uniform bound on the approximation of bounded function $f = (f_1, f_2, ..., f_p) : [0,1]^p \to \mathbb{R}^q$ by Bernstein polynomial $\widetilde{B}_n(f, \cdot) : [0,1]^p \to \mathbb{R}^q$ which is defined by

$$\widetilde{B}_n(f, \cdot) = (B_n(f_1, \cdot), B_n(f_2, \cdot), \ldots, B_n(f_q, \cdot))$$

where $B_n(f_k, \cdot) : [0,1]^p \to \mathbb{R}$ define by

$$\begin{aligned} &B_n(f_k, \vec{t}) \\ &= \sum_{j_1+j_2+\ldots+j_{p+1}=n} f_k(\frac{pj_1}{n}, \frac{pj_2}{n}, \ldots, \frac{pj_p}{n}) \binom{n}{j_1, j_2, \ldots, j_{p+1}} T_1^{j_1} T_2^{j_2} \ldots T_{p+1}^{j_{p+1}} \end{aligned} \tag{1.1}$$

for $\vec{t} = (t_1, t_2, ..., t_p)$, $T_i = \dfrac{t_i}{p}, i = 1, 2, ..., p, T_{p+1} = 1 - (T_1 + T_2 + ... + T_p)$ and $\binom{n}{j_1, j_2, \ldots, j_{p+1}} = \dfrac{n!}{j_1! j_2! \ldots j_{p+1}!}$.
Observe that, in case of $p = q = 1$,

$$\widetilde{B}_n(f, t) = \sum_{j=0}^{n} \binom{n}{j} t^j (1-t)^{n-j} f(\frac{j}{n}) \tag{1.2}$$

is the Bernstein polynomial function which defined by Bernstein([8]). We note that Bernstein polynomial are useful in Bayesian statistics because of their interpretation as mixtures of Beta distribution ([6], [9]). The original work is back to 1912, when Bernstein ([8]) gave the well-known Weierstrass approximation theorem.

Theorem 1.1. *(Weierstrass Approximation Theorem)*
Let $f : [0, 1] \to \mathbb{R}$ be a continuous function and $\widetilde{B}_n(f, \cdot) : [0, 1] \to \mathbb{R}$ be Bernstein polynomial which defined by (1.2). Then for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\left| f(t) - \widetilde{B}_n(f, t) \right| < \epsilon$$

for every $t \in [0, 1]$.

Since then, there are many authors investigate a bound of this approximation such as Popoviciu ([7]) , 1934 , Kac ([4],[5]) in 1938 - 1939 , Dallal and Hall ([9]) in 1983 , Diaconics and Ylvisaker ([6]) in 1985.

Until 1997 Both Gzyl and Palacios ([1]) investigated a bound of the approximation in case of f is Lipschitz function i.e., there exists a constant L such that $|f(x) - f(y)| \le L|x - y|$ for any $x, y \in [0, 1]$. They yield the rate $\sqrt{\dfrac{\ln n}{n}}$ as follows: $\left| f(t) - \widetilde{B}_n(f, t) \right| \le K\sqrt{\frac{\ln n}{n}}$ for all $t \in [0, 1]$.

In case of multi-dimension, K.Neammanee and S.Sirisub([3]) used the probalistic tools to approximate and they showed that for $\epsilon > 0$,

$$\left\| f(\vec{t}) - \widetilde{B}_n(f, \vec{t}) \right\| \le \epsilon \text{ for all } \vec{t} \in [0, 1]^p. \tag{1.3}$$

In (1.3) K.Neammanee and S.Sirisub, they did not give bound of the approximation. In 2001, K.Neammanee([2]) gave a bound of this approximation in case of Lipschitz function. He showed that

$$\left\|f(\vec{t}) - \widetilde{B}_n(f, \vec{t})\right\| \leq K\sqrt{\frac{\ln n}{n}} \quad \text{for} \quad \vec{t} \in [0,1]^p. \tag{1.4}$$

A bound in (1.4) was improved in 2004 by Y.Pankla and E.Suntonsinsongvon([11]). Let $f : \mathbb{R}^p \to \mathbb{R}^q$ be holder function with exponent $\alpha > 0$, i.e., there exists a constant C such that $\|f(x) - f(y)\| \leq C\|x - y\|^\alpha$ for every $x, y \in [0,1]^p$. Then there exists a constant K depend on f such that

$$\left\|f(\vec{t}) - \widetilde{B}_n(f, \vec{t})\right\| \leq \frac{K}{n^{\frac{\alpha}{2}}} \quad \text{for} \quad \vec{t} \in [0,1]^p \text{ and } n \in \mathbb{N}. \tag{1.5}$$

In this paper we will relieve the condition of Y.Punkla and E.Sontonsinsongvon to a bounded function. Theorem 3 is our main result.

Theorem 1.2. *Let $f : [0,1]^p \to \mathbb{R}^q$ be a bounded function, then there exists a constant $C > 0$ such that*

$$\left\|f(\vec{t}) - \widetilde{B}_n(f, t)\right\| \leq C\omega(\frac{1}{\sqrt{n}}) \quad \textit{for all} \quad t \in [0,1]^p$$

where the modulus $\omega(\delta)$ of f is defined for $\delta > 0$ by

$$\omega(\delta) = \sup_{\substack{\vec{t_1}, \vec{t_2} \in [0,1]^p \\ \left\|\vec{t_1} - \vec{t_2}\right\| \leq \delta}} \left\|f(\vec{t_1}) - f(\vec{t_2})\right\|.$$

Note that the modulus of f depends on δ, f and the interval $[0,1]^p$. So that $\omega(\delta)$ is shorthand for $\omega(f; [0,1]^p; \delta)$. It f is holder with exponent $\alpha > 0$, we have $\omega(\frac{1}{\sqrt{n}}) = \frac{C}{n^{\alpha/2}}$. Hence Theorem 1.2 generizes (1.5).

2 Proof of Main Results

By the fact that

$$\begin{aligned}
&\left\|f(\overrightarrow{t})-\widetilde{B}_n(f,\overrightarrow{t})\right\| \\
&=\left\|(f_1(\overrightarrow{t}),f_2(\overrightarrow{t}),\dots,f_p(\overrightarrow{t}))-(B_n(f_1,\overrightarrow{t}),B_n(f_2,\overrightarrow{t}),\dots,B_n(f_p,\overrightarrow{t}))\right\| \\
&=\sqrt{(f_1(\overrightarrow{t})-B_n(f_1,\overrightarrow{t}))^2+(f_2(\overrightarrow{t})-B_n(f_2,\overrightarrow{t}))^2+\ldots+(f_p(\overrightarrow{t})-B_n(f_p,\overrightarrow{t}))^2} \\
&\le\sqrt{\left(\left|f_1(\overrightarrow{t})-B_n(f_1,\overrightarrow{t})\right|+\left|f_2(\overrightarrow{t})-B_n(f_2,\overrightarrow{t})\right|+\ldots+\left|f_p(\overrightarrow{t})-B_n(f_p,\overrightarrow{t})\right|\right)^2} \\
&=\sum_{k=1}^{q}\left|f_k(\overrightarrow{t})-B_n(f_k,\overrightarrow{t})\right|, \qquad (2.1)
\end{aligned}$$

it suffices to prove that for each $k=1,2,...,p$, there exists a positive constant C such that

$$\left|f_k(\overrightarrow{t})-B_n(f_k,\overrightarrow{t})\right|\le C\omega(\frac{1}{\sqrt{n}}).$$

By the fact that

$$1=\sum_{j_1+j_2+\ldots+j_{p+1}=n}\binom{n}{j_1,j_2,\dots,j_{p+1}}T_1^{j_1}T_2^{j_2}\dots T_{p+1}^{j_{p+1}}$$

for $\overrightarrow{t}=(t_1,t_2,...,t_p)\in[0,1]^p, T_i=\frac{t_i}{p}, i=1,2,...,p$ and $T_{p+1}=1-(T_1+T_2+...+T_p)$(eq.(1) of [11]), we have

$$f_k(\overrightarrow{t})=\sum_{j_1+j_2+\ldots+j_{p+1}=n}f_k(\overrightarrow{t})\binom{n}{j_1,j_2,\dots,j_{p+1}}T_1^{j_1}T_2^{j_2}\dots T_{p+1}^{j_{p+1}}.$$

Hence

$$\begin{aligned}
&\left|f_k(\overrightarrow{t})-B_n(f_k,\overrightarrow{t})\right| \\
&\quad=\left|\sum_{j_1+j_2+\ldots+j_{p+1}=n}\left(f_k(\overrightarrow{t})-f_k(\frac{pj_1}{n},\frac{pj_2}{n},\dots,\frac{pj_p}{n})\right)\binom{n}{j_1,j_2,\dots,j_{p+1}}T_1^{j_1}T_2^{j_2}\dots T_{p+1}^{j_{p+1}}\right| \\
&\quad\le\sum_{j_1+j_2+\ldots+j_{p+1}=n}\left|\left(f_k(\overrightarrow{t})-f_k(\frac{pj_1}{n},\frac{pj_2}{n},\dots,\frac{pj_p}{n})\right)\right|\binom{n}{j_1,j_2,\dots,j_{p+1}}T_1^{j_1}T_2^{j_2}\dots T_{p+1}^{j_{p+1}} \\
&\quad=\sum_{j_1+j_2+\ldots+j_{p+1}=n}\omega(\left\|\overrightarrow{t}-(\frac{pj_1}{n},\frac{pj_2}{n},\dots,\frac{pj_p}{n})\right\|)\binom{n}{j_1,j_2,\dots,j_{p+1}}T_1^{j_1}T_2^{j_2}\dots T_{p+1}^{j_{p+1}}.
\end{aligned} \qquad (2.2)$$

Note that

$$\sum_{j_1+j_2+...+j_{p+1}=n} j_i \binom{n}{j_1, j_2, ..., j_{p+1}} T_1^{j_1} T_2^{j_2} ... T_{p+1}^{j_{p+1}} = nT_i$$

and

$$\sum_{j_1+j_2+...+j_{p+1}=n} j_i^2 \binom{n}{j_1, j_2, ..., j_{p+1}} T_1^{j_1} T_2^{j_2} ... T_{p+1}^{j_{p+1}} = n(n-1)T_i^2 + nT_i$$

for $i = 1, 2, ..., p+1$(eq.(4) and (5) of [11]).

Hence

$$\sum_{j_1+j_2+...+j_{p+1}=n} \left\| \vec{t} - (\frac{pj_1}{n}, \frac{pj_2}{n}, \dots, \frac{pj_p}{n}) \right\| \binom{n}{j_1, j_2, \dots, j_{p+1}} T_1^{j_1} T_2^{j_2} \dots T_{p+1}^{j_{p+1}}$$

$$\leq \Big[\sum_{j_1+j_2+...+j_{p+1}=n} \left\| \vec{t} - (\frac{pj_1}{n}, \frac{pj_2}{n}, \dots, \frac{pj_p}{n}) \right\|^2 \binom{n}{j_1, j_2, \dots, j_{p+1}} T_1^{j_1} T_2^{j_2} \dots T_{p+1}^{j_{p+1}} \Big]^{\frac{1}{2}}$$

$$\Big[\sum_{j_1+j_2+...+j_{p+1}=n} \binom{n}{j_1, j_2, \dots, j_{p+1}} T_1^{j_1} T_2^{j_2} \dots T_{p+1}^{j_{p+1}}]^{\frac{1}{2}} \Big]^{\frac{1}{2}}$$

$$= \Big[\sum_{j_1+j_2+...+j_{p+1}=n} \left\| \vec{t} - (\frac{pj_1}{n}, \frac{pj_2}{n}, \dots, \frac{pj_p}{n}) \right\|^2 \binom{n}{j_1, j_2, \dots, j_{p+1}} T_1^{j_1} T_2^{j_2} \dots T_{p+1}^{j_{p+1}} \Big]^{\frac{1}{2}}$$

$$= \Big[\sum_{i=1}^{p} \sum_{j_1+j_2+...+j_{p+1}=n} (t_i - \frac{pj_i}{n})^2 \binom{n}{j_1, j_2, \dots, j_{p+1}} T_1^{j_1} T_2^{j_2} \dots T_{p+1}^{j_{p+1}} \Big]^{\frac{1}{2}}$$

$$= \Big[\sum_{i=1}^{p} \sum_{j_1+j_2+...+j_{p+1}=n} [p(T_i - \frac{j_i}{n})]^2 \binom{n}{j_1, j_2, \dots, j_{p+1}} T_1^{j_1} T_2^{j_2} \dots T_{p+1}^{j_{p+1}} \Big]^{\frac{1}{2}}$$

$$= p\Big[\sum_{i=1}^{p} \sum_{j_1+j_2+...+j_{p+1}=n} (T_i^2 - \frac{2j_iT_i}{n} + \frac{j_i^2}{n^2}) \binom{n}{j_1, j_2, \dots, j_{p+1}} T_1^{j_1} T_2^{j_2} \dots T_{p+1}^{j_{p+1}} \Big]^{\frac{1}{2}}$$

$$= \frac{p}{n^{\frac{1}{2}}} \Big[\sum_{i=1}^{p} T_i(1-T_i) \Big]^{\frac{1}{2}}$$

$$\leq \frac{C}{n^{1/2}}. \tag{2.3}$$

By the fact that $\omega(\lambda\delta) \leq (1+\lambda)\omega(\delta)$ for $\lambda > 0$([10],pp.15), we have

$$\omega(\left\| \vec{t} - (\frac{pj_1}{n}, \frac{pj_2}{n}, \dots, \frac{pj_p}{n}) \right\|) = \omega(n^{\frac{1}{2}} \left\| \vec{t} - (\frac{pj_1}{n}, \frac{pj_2}{n}, \dots, \frac{pj_p}{n}) \right\| n^{\frac{-1}{2}})$$

$$= (1 + n^{\frac{1}{2}} \left\| \vec{t} - (\frac{pj_1}{n}, \frac{pj_2}{n}, \dots, \frac{pj_p}{n}) \right\|) \omega(\frac{1}{\sqrt{n}}). \tag{2.4}$$

Therefore, by (2.2)-(2.4),

$$
\begin{aligned}
&|f_k(\vec{t}) - B_{nk}(\vec{t})| \\
&\leq \omega(\frac{1}{\sqrt{n}})[\sum_{j_1+j_2+\ldots+j_{p+1}=n} \binom{n}{j_1, j_2, \ldots, j_{p+1}} T_1^{j_1} T_2^{j_2} \ldots T_{p+1}^{j_{p+1}} \\
&\quad + n^{\frac{1}{2}} \sum_{j_1+j_2+\ldots+j_{p+1}=n} \left\| \vec{t} - (\frac{pj_1}{n}, \frac{pj_2}{n}, \ldots, \frac{pj_p}{n}) \right\| \binom{n}{j_1, j_2, \ldots, j_{p+1}} T_1^{j_1} T_2^{j_2} \ldots T_{p+1}^{j_{p+1}}] \\
&\leq C\omega(\frac{1}{\sqrt{n}}).
\end{aligned}
$$

Then the theorem is proved. #

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