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Cohomological Properties of the Restriction of $S^m(\bigwedge^k(T\mathbf{P}^n))$ to Projective Curves

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Abstract. Here we prove that $h^i(C, S^m(\bigwedge^k(T\mathbf{P}^n))(t)|C)(-A))$, $i = 1$ or $i = 0$ for several curves $C \subset \mathbf{P}^n$ and for general finite subsets of C with a prescribed cardinality

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1. Introduction

Fix integers n, d, g such that $n \geq 3$, $g \geq 0$ and either $d \geq g+3$ or $d \geq 2n+1$ and $d - n < g \leq d - n + \lfloor (d-n)/(n+1) \rfloor$. In [2] (case $n = 3$) and [3] (case $n > 3$) the authors introduced and studied a very nice irreducible component $Z(d, g; n)$ of the Hilbert scheme $\mathrm{Hilb}(\mathbf{P}^n)$ of all curves in $\mathbf{P}^n$ with degree d and genus g. $Z(d, g; n)$ is generically smooth, $\dim(Z(d, g; n)) = (n+1)d + (n-3)(1-g)$, a general $C \in Z(d, g; n)$ is smooth, $h^1(C, N_C) = 0$, $h^1(C, \mathcal{O}_C(2)) = 0$. Furthermore, if $d \geq g + n$, then $h^1(C, \mathcal{O}_C(1)) = 0$ and if $C \leq d + n$, then C is linearly normal. If $g \geq 1$, $d \geq 2n+1$ and $n+2)\lfloor (d-n-1)/n \rfloor$, then $T\mathbf{P}^n|C$ is semistable ([4], Prop. 1.3). If $g \geq 2$, $d \geq 3n+2$ and $(n+2)\lfloor (d-2n-1)/n \rfloor$, then $T\mathbf{P}^n|C$ is stable.

Theorem 1. *Fix integers n, d, g, k, m such that $n \geq 3$, $1 \leq k \leq n-1$, $m \geq 1$, $t \geq 0$, $g \geq 0$, $d \geq 2n$ and either $d \geq g + n$ or $g - d - n - 2 \leq \lfloor d/n \rfloor$. If either $d \equiv 0 \mod n$ or $m = k = 1$ and $g > 0$ or $m = 1$, $k = n-1$ and $g > 0$, then set $\epsilon_{d,g,n,k,m} := 0$. In all other cases set $\epsilon_{d,g,n,k,m} := mk'$, where $k' := \min\{k, n-k\}$.*

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Set $\alpha := km(n+1)d + \binom{\binom{n}{k}+m-1}{m}\cdot(t+1-g)$*. Fix a general* $C \in Z(d,g,n)$ *and general finite sets* $A \subset C$*,* $B \subset C$ *such that* $\sharp(A) \le \lfloor \alpha/\binom{\binom{n}{k}+m-1}{m}\rfloor - \epsilon_{d,g,n,k,m}$ *and* $\sharp(B) \ge \lceil \alpha/\binom{\binom{n}{k}+m-1}{m}\rceil + \epsilon_{d,g,n,k,m}$*. Then* $h^0(C, S^m(\bigwedge^k(T\mathbf{P}^n))(t)|C) = \alpha$*,* $h^1(C, S^m(\bigwedge^k(T\mathbf{P}^n))(t)|C)) = 0$*,* $h^1(C,(S^m(\bigwedge^k(T\mathbf{P}^n))(t)|C(-A)) = 0$ *and* $h^0(C,(S^m(\bigwedge^k(T\mathbf{P}^n))(t)|C(-B)) = 0$.

We leave to the interested reader to follow the proof of Theorem 1 to extend its stament to the composition of other Schur functors of $T\mathbf{P}^n$, not just the symmetric and alternating powers.

We work over an algebraically closed field $\mathbb{K}$.

Lemma 1. *Let* Y *a reduced equidimensional projective curve such that* $Y = C \cup D$ *with* $C \ne \emptyset$ *and* $D \ne \emptyset$*. Set* $Z := C \cap D$ *(scheme-theoretic intersection). Let* F *be a vector bundle on* Y *and* $A \subset C \backslash Z_{red}$*,* $B \subset D\backslash Z_{red}$ *closed subschemes such that* $h^1(C, \mathcal{I}_{A,C} \otimes (F|C)) = h^1(D, \mathcal{I}_{B\cup Z,D} \otimes (F|D)) = 0$*. Then* $h^1(Y, \mathcal{I}_{A\cup B,Y} \otimes F) = 0$*.*

Proof. Since $A_{red} \cap Z_{red} = B_{red} \cap Z_{red} = \emptyset$ and F is locally free, we have a Mayer-Vietoris exact sequence on Y:

$$0 \to \mathcal{I}_{A\cup B,Y} \otimes F \to \mathcal{I}_{A,C} \otimes (F|C) \oplus \mathcal{I}_{B,D} \otimes (F|D) \to F|Z \to 0 \tag{1}$$

Since $h^1(D, \mathcal{I}_{B\cup Z,D} \otimes (F|D)) = 0$, the restiction map $H^0(D, \mathcal{I}_{B,D} \otimes (F|D)) \to H^0(Z, F|Z)$ is surjective. Apply the cohomology exact sequence associated to (1). $\square$

Remark 1. In the set-up of the statement of Lemma 1 we have $h^1(D, \mathcal{I}_{B\cup Z,D} \otimes (F|D)) = 0$ if $D \cong \mathbf{P}^1$ and $F|D$ has splitting type $a_1 \ge \cdots \ge a_n$ with $a_n \ge \text{length}(Z \cup B) - 1$.

Taking $A = B = \emptyset$ in the proof of Lemma 1 we get the following result.

Lemma 2. *Let* Y *a reduced equidimensional projective curve such that* $Y = C \cup D$ *with* $C \ne \emptyset$ *and* $D \ne \emptyset$*. Set* $Z := C \cap D$ *(scheme-theoretic intersection). Let* F *be a vector bundle on* Y *such that the restriction map* $H^0(D, F|D) \to H^0(Z, F|Z)$ *is surjective. Then:*

$$h^0(Y, F) = h^0(C, F|C) + h^0(D, F|D) - \mathit{length}(Z) \cdot \mathit{rank}(F) \tag{2}$$

$$h^1(Y, F) = h^1(C, F|C) + h^1(D, F|D) \tag{3}$$

Remark 2. Let $D \subset \mathbf{P}^n$ be a rational normal curve. Then $T\mathbf{P}^n|D$ is isomorphic to the direct sum of n line bundles of degree $n + 1$ (see e.g. [4], Lemma 1.3).

Remark 3. Fix an integer $d \ge g + 1$ and let $C \subset \mathbf{P}^n$ be a general $C \in Z(d, 1; n)$. Then $T\mathbf{P}^n|C$ is semistable ([4], Prop. 1.4). We will also need that by Atiyah's classification of vector bundles on elliptic curves ([1], Part III) even in positive charateristic the wedge product and the symmetric product of a semistable vector bundle on C is semistable.

Remark 4. Fix integers $n > k \geq 1$, $m \geq 1$, a reduced and connected projective curve Y, $M \in \mathrm{Pic}(Y)$ and a rank n vector bundle F on Y. Set $x := \deg(F)$ and $y := \deg(M)$; here deg means the total degree. Notice that $\mathrm{rank}(\bigwedge^k(F)) = \binom{n}{k}$ and $\mathrm{rank}(S^m(\bigwedge^k(F))) = \binom{\binom{n}{k}+m-1}{m}$. By Riemann-Roch we have $\chi(S^m(\bigwedge^k(F)) \otimes M) = \deg(S^m(\bigwedge^k(F)) \otimes M) + \binom{\binom{n}{k}+m-1}{m}(1-p_a(Y))$. We have $\deg(S^m(\bigwedge^k(F)) \otimes M) = \deg(S^m(\bigwedge^k(F))) + y \cdot \binom{\binom{n}{k}+m-1}{m}$ ([5], Lemma 2.1). For any vector bundle A on Y let $\mu(A) := \deg(A)/\mathrm{rank}(A)$ denote the slope of A. We have $\mu(A \otimes B) = \mu(A) + \mu(B)$ for all vector bundles A, B. Since in characteristic zero $\bigwedge^k(A)$ (resp. $S^m(A)$) is a direct summand of $A^{\otimes k}$ (resp. $A^{\otimes m}$, in characteristic zero we get $\mu(S^m(\bigwedge^k(F))) = km \cdot \mu(F) = kmx/n$. Using the splitting principle we get that the same equality is true in positive characteristic. Thus $\deg(S^m(\bigwedge^k(F))) = km\binom{\binom{n}{k}+m-1}{m}/n$.

Proof of Theorem 1. First assume $k = m = 1$ and $g = 0$. Let e the only integer such that $n \leq e \leq 2n-1$ and $d \equiv n \mod n$. Set $d = un + e$ with $u \geq 0$. Take a general $T \in Z(e, 0; n)$. By [6] $T\mathbf{P}^n|T$ is rigid, i.e. its splitting type $a_1 \geq \cdots \geq a_n$ satisfies $a_n \geq a_1 - 1$. Notice that $a_n = a_1$ if $e = 0$ (Remark 2). In all other cases we have $\epsilon_{d,g,n,1,1} = 1$ and we are allowed at this step to loose one condition: we need it because $h^0(C, (T\mathbf{P}^n|T)(-B)) = 0$ if and only if $\sharp(B) \geq a_1$, while $h^1(C, (T\mathbf{P}^n|T)(-A)) = 0$ if and only if $\sharp(A) \leq a_{n-1}+1$. Then we apply u times Lemma 2 taking as the second curve a rational normal curve intersecting the first curve at exactly one point. The same proof works even if $k = n-1$, $m = 1$ and $g = 0$. If either $k \neq \{1, n-1\}$ or $m \geq 2$, then we just use at the first step the value for $\epsilon d, g, n, k, m$ and then apply u times Lemma 2 to $S^m(\bigwedge^k(T\mathbf{P}^n))(t)$ with a rational normal curve as the second curve. Of course, we also use Remark 4. Now assume $g > 0$. Let f be the only integer such that $n + 1 \leq f \leq 2n$ and $d \equiv f \pmod n$. Set $d = f + vn$. Now we start with a general $M_0 \in Z(f, 1; n)$ to which we apply Remark 3. Then we add u times a rational normal curve D_i, $1 \leq i \leq v$, so that each curve $M_i := M_0 \cup D_1 \cup \cdots D_i$, $1 \leq i \leq v$, is connected and nodal and $1 \leq \sharp(M_{i-1} \cap D_i) \leq n+1$. We met in this way exactly the prescribed general pairs (d, g). □

References

[1] M. Atiyah, Vector bundles over an elliptic curve, Proc. London Math. Soc. (3) 7 (1957), 414–452; reprinted in: Michael Atiyah Collected Works, Vol. 1, 105–143, Oxford, 1988.

[2] E. Ballico and Ph. Ellia, Beyond the maximal rank conjecture for curves in $\mathbb{P}^3$, in: Space Curves, Proc. Rocca di Papa 1985, Lect. Notes in Math. 1266, Springer, Berlin, 1987.

[3] E. Ballico and Ph. Ellia, On the existence of curves with maximal rank in $\mathbb{P}^3$, J. Reine Angew. Math. 397 (1989), 1–22.
[4] E. Ballico and G. Hein, On the stability of the restriction of $T\mathbf{P}^n$ to projective curves, Arch. Math (Basel) 71 (1998), no. 1, 80–88.
[5] R. Hartshorne, Stable reflexive sheaves, Math. Ann. 254 (1980), no. 2, 121–176.
[6] L. Ramella, Sur les schémas de Hilbert des courbes rationnelles lisses de $\mathbf{P}^n$ par le fibré tangent restreint, C. R. Acad. Sci. Paris, Sér. I Math. 311 (1990), 181–184.

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