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Spectral Theory of Ordered Pairs of the Linear Operators - acting in Different Banach Spaces and Applications[1]

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Abstract. Ordered pairs of linear operators (A, B) from Banach space $L(X; Y)$ of the linear bounded operators, defined on complex Banach space X with the values in complex Banach space Y are considered in this paper.

Notion of singular set (spectrum) of the ordered pair (A, B) is introduced essentially with the help of operators bundle of the form

$$A - \lambda B,\ \ \lambda \in \mathbb{C},$$

and functional calculus in sections (1,2). For investigation of spectral properties of the pairs (A, B) we apply left and right pseudoresolvents of the pair (A, B), which allow to apply theory of commutative Banach algebras and spectral theory of the operators, acting in one space. In sections (1,2) we give the number of general results on the spectrum of operators pairs (A, B) in dependence on some properties of the operator B (connected with invertibility, finite dimensionless and etc.).

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1. Spectrum and Regular Set of the Ordered Pairs of the Operators

First of all we note more applied notations. They are denoted by $\sigma(T)$ and $\rho(T)$ the spectrum and resolvent set of the linear operator $T : D(T) \subset Z \to Z$ ($D(T)$ is its domain), acting in complex Banach space Z.

Definition 1. *Spectrum of the pair (A, B), where $A, B \in L(X; Y)$ is the collection of such complex $\lambda \in C$, for which the operator $A - \lambda B$ has not bounded inverse.*

Spectrum of the pair (A, B) we denote by symbol $\sigma(A, B)$. Denote by $\rho(A, B)$ the set $\mathbb{C}/\sigma(A, B)$. It is clear that the spectrum of the pair (A, B) is closed, and the set $\rho(A, B)$ is open. In the case of difference of the spectrum of bounded operator (acting in one space), spectrum of the pair (A, B) can be unbounded set. Moreover, it may be empty.

Example 1. *We consider closed linear operator*

$$A : D(A) \subset Y \to Y$$

acting in one Banach space Y with the domain $D(A)$. In case of $X = D(A)$ and we norm the space X, supposing $\|x\|_* = \|x\| + \|Ax\| \quad \forall x \in X$ (In this case X becomes Banach space). Let $Bx = x$, $x \in X$, $B : X \to Y$ the operator of embedding operator. Then it is clear that $\sigma(A, B)$ coincide with usual spectrum $\sigma(A)$ of the operator $A : D(A) \subset Y \to Y$. Thus, above-mentioned possibilities arise for singular set $\sigma(A, B)$ (for example, if $\sigma(A) = \emptyset$, then also $\sigma(A, B) = \emptyset$). Possible emptiness of the spectrum $\sigma(A, B)$ comes to definite difficulties and in this connection we give notion of extended spectrum of the pair (A, B).

Definition 2. *Extended spectrum $\tilde{\sigma}(A, B)$ of the pair (A, B) is a subset from the extended complex plane $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, which coincides with $\sigma(A, B)$, if both functions $\lambda \to B(A - \lambda B)^{-1} : \tilde{\mathbb{C}} \to L(Y)$, $\lambda \to (A - \lambda B)^{-1} B : \tilde{\mathbb{C}} \to L(X)$ are halomorphic at the point ∞ and coincides with $\sigma(A, B) \cup \{\infty\}$ in opposite case. Suppose $\tilde{\rho}(A, B) = \tilde{\mathbb{C}}/\tilde{\sigma}(A, B)$.*

The function $R_{A,B}(\cdot) = R(\cdot, A, B) : \rho(A, B) \to L(Y; X)$ defined by the equality.

$$R_{A,B}(\lambda) = (A - \lambda B)^{-1}, \ \lambda \in \rho(A, B) \tag{1.1}$$

is called the resolvent of operators pair (A, B) from $L(X; Y)$. From here, it is clear that

$$R(\lambda;\ A, B) - R(\mu;\ A, B) = (\mu - \lambda) R(\lambda;\ A, B) B R(\mu;\ A, B).$$

From this equality particularly, we obtain for any $\lambda, \mu \in \rho(A, B)$ the following identities

$$\begin{aligned} R_l(\lambda) - R_l(\mu) &= (\mu - \lambda) R_l(\lambda) R_l(\mu), \\ R_r(\lambda) - R_r(\mu) &= (\mu - \lambda) R_r(\lambda) R_r(\mu), \end{aligned}$$

where the functions

$$R_l(\lambda) = BR(\lambda; A, B),\, R_r(\lambda) = R(\lambda; A, B)\, B,\ \lambda \in \rho(A, B) \tag{1.2}$$

are called the left and right pseudoresolvents for the pair (A, B). *Note that the functions* $R_l : \rho(A, B) \to L(Y)$, $R_r : \rho(A, B) \to L(X)$ *are pseudoresolvents in sense* [1] *and introduced terminology is connected with it. (Note, that the function* $R : U \to L(Z)$, *defined on the opened set* $U \subset \mathbb{C}$ *is called pseudoresolvent, if* $\lambda, \mu \in U$ *and the following identity* $R(\lambda) - R(\mu) = (\mu - \lambda) R(\lambda) R(\mu)$ *takes place.*

In the content of this study (without special reminding) non-emptiness of regular set of the considered ordered pairs of the operators from $L(X; Y)$ *is supposed.*

Lemma 1. *Left and right pseudoresolvents* R_l *and* R_r *of the pair* (A, B) *are the resolvent of some operators if and only if the operator* B *is invertible (not necessary continuous).*

Proof. Let $B : X \to Y$ be invertible operator and ImB be its values set. Then the operator AB^{-1} has the domain $D(AB^{-1})$, equals to ImB and acts in one space Y. Let $\lambda_0 \in \rho(A, B)$. Then

$$R_l(\lambda_0)\left(AB^{-1} - \lambda_0 I_Y\right) y = B(A - \lambda_0 B)^{-1} (A - \lambda_0 B) B^{-1} y = y$$

for any $y \in ImB = D(AB^{-1})$ and

$$\left(AB^{-1} - \lambda_0 I_Y\right) R_l(\lambda_0) = (A - \lambda_0 B) B^{-1} B (A - \lambda_0 B)^{-1} = I_Y.$$

From these equalities it follows that $Re(\lambda) = \left(AB^{-1} - \lambda I_Y\right)^{-1},\ \lambda \in \rho(A, B)$, i.e. R_l is the resolvent of the operator AB^{-1}. Now let's consider the operator $C = B^{-1}A : D(C) \subset X \to X$ with domain $D(C) = \{x \in X : Ax \in ImB\}$. Apriority it is not clear, why $D(C) \neq \{0\}$. However the condition $\rho(A, B) \neq \emptyset$, as we'll be convinced now, guarantees that $\lambda_0 \in \rho(A, B) \neq \emptyset$. Let $\lambda_0 \in \rho(A, B)$. Then the operator $R_r(\lambda_0) = (A - \lambda_0 B)^{-1} B$ has the property $R_r(\lambda_0) x \in D(C)$, $\forall x \in X$ and, moreover, the set of values of this operator coincides with $D(C)$. Really, it follows from the equalities it also follows that

$$\begin{aligned} AR_r(\lambda_0) &= A(A - \lambda_0 B)^{-1} B = (A - \lambda_0 B + \lambda_0 B)(A - \lambda_0 B)^{-1} B \\ &= B + \lambda_0((A - \lambda_0 B)^{-1} B = B(I_X + \lambda_0 R_r(\lambda_0)) \\ \left(B^{-1}A - \lambda_0 I_X\right) R_r(\lambda_0) &= I_X \end{aligned}$$

and

$$R_r(\lambda_0)\left(B^{-1}A - \lambda_0 I_X\right) x = x,\ \forall x \in D(C).$$

∎

Lemma is proved.

Directly from the Lemma 1 and its proof we obtain two following corollaries.

Corollary 2. *If the operator B is invertible (not necessary continuously), then*

$$\sigma(A,B) = \sigma\left(AB^{-1}\right) \cup \sigma\left(B^{-1}A\right).$$

Corollary 3. *If subspace*

$$D(C) = \{x \in X : Ax \in ImB\}$$

consists only of zero vector and $X \neq \{0\}$ *then* $\sigma(A,B) = C$. *Moreover,* $\sigma(A,B) = C$, *if* $D(C)$ *is finite-dimensional subspace from infinite-dimensional space* X.

Note that if $\rho(A,B) \neq \oslash$, *then for* $\lambda_0 \in \rho(A,B)$ *the operator* $B^{-1}A - \lambda_0 I_X : D(C) \subset X \to X$ *will be invertible. It is not possible because of finite-dimensionness of* $D(C)$ *and infinite-dimensionness of* X.

The case if B is non-invertible operator will be considered in the next paragraph.

2. **Operator Calculus**

Consider the pair (A,B) and let Δ be open set from the extended complex plane $\tilde{\mathbb{C}}$, containing the extended spectrum $\tilde{\sigma}(A,B)$ of this pair. Denoted by symbol $H(\Delta)$ the topological algebra of halomorphic on Δ functions with topology of uniform convergent on compact subsets from Δ. Always denote by γ oriented Jordan contour, not crossing the point ∞ and such that $\tilde{\sigma}(A,B)$ belongs to its inside part.

For each function $f \in H(\Delta)$ we suppose

$$T(f) = \frac{1}{2\pi}\int_{\gamma} f(\lambda)(A-\lambda B)^{-1}\,d\lambda \tag{2.1}$$

in supposition that the point ∞ is out of the contour γ. Formula (2.1) defines the continuous linear operator

$$T : H(\Delta) \to L(Y,X).$$

Lemma 4. *If* $\lambda_0 \in \rho(A,B)$, *then the operator*

$$T(f_0) = \frac{1}{2\pi i}\int_{\gamma} \frac{1}{\lambda-\lambda_0}(A-\lambda B)^{-1}\,d\lambda, \quad f_0(\lambda) = \frac{1}{\lambda-\lambda_0}$$

which coincides with the operator $(A-\lambda_0 B)^{-1}$.

Proof. The equalities

$$\begin{aligned}(A-\lambda_0 B)\,T(f_0) &= \frac{1}{2\pi i}(A-\lambda_0 B)\int\limits_\gamma \frac{1}{\lambda-\lambda_0}(A-\lambda B)^{-1}\,d\lambda\\ &= \frac{1}{2\pi i}\int\limits_\gamma \frac{[A-\lambda B+(\lambda-\lambda_0)\,B]\,(A-\lambda B)^{-1}}{\lambda-\lambda_0}d\lambda\\ &= \frac{1}{2\pi i}\int\limits_\gamma \frac{d\lambda}{\lambda-\lambda_0}I_y+\frac{1}{2\pi i}\int\limits_\gamma B\,(A-\lambda B)^{-1}\,d\lambda = I_y\end{aligned}$$

take place. ∎

Similarly the equality $T(f_0)(A-\lambda_0 B) = I_X$ is proved. Lemma is proved. In order to solve the difficulties, connected with the fact that $L(Y;X)$ is not algebra for $L(Y;X)$, we introduce two linear operators:

$$T_l : H(\Delta) \to L(Y)\,;\ T_r : H(\Delta) \to L(X)\,,$$

defined for any function $f \in H(\Delta)$ by the formulas

$$T_l(f) = \delta f(\infty)\,I_Y + \frac{1}{2\pi i}\int\limits_\gamma f(\lambda)\,R_l(\lambda)\,d\lambda, \tag{2.2}$$

$$T_r(f) = \delta f(\infty)\,I_X + \frac{1}{2\pi i}\int\limits_\gamma f(\lambda)\,R_r(\lambda)\,d\lambda, \tag{2.3}$$

where $\delta = 0$ or $\delta = 1$ without considering the point ∞ whether inside γ or outside γ. Note one more following simple connection of the operators

$$T_r(f) = T(f)\,B;\ T_l(f) = BT(f)\,,\ f \in H(\Delta) \tag{2.4}$$

and it is supposed that in corresponding formulas (2.2)–(2.3) contour γ is chosen such that the point ∞ is out of γ (this supposition is done, because the formula (2.1) is defined only for such contours). As R_l and R_r are pseudoresolvents, then using usual reasonings in the proof of properties of the operator calculus we obtain that it takes place.

Lemma 5. *Operators $T_r : H(\Delta) \to L(X)\,,\ T_l : H(\Delta) \to L(Y)$ are continuous homomorphisms of algebra $H(\Delta)$ and the following equalities*

$$T_r(f)\,T(\varphi) = T(f)\,BT(\varphi) = T(\varphi)\,BT(f) = T(\varphi)\,T_l(f) \tag{2.5}$$

$$T_l(f)\,(A-\mu B) = (A-\mu B)\,T_r(f) \tag{2.6}$$

$$T_l(f_0) = BR(\lambda_0;A,B)\,,\ T_r(f_0) = R(\lambda_0;A,B)\,B, \tag{2.7}$$

take place for any $f,\varphi \in H(\Delta)\,,\ \lambda_0 \in \rho(A,B)$ and $f_0 = \frac{1}{\lambda-\lambda_0}$. Note that the equality (2.7) follows from the equalities (2.4) and lemma 1. We'll prove the *equality (2.6). It follows from the following equalities*

$$\begin{aligned}(A-\mu_0 B)\,T_r(f) &= \frac{1}{2\pi i}\int_\gamma (A-\mu_0 B)f(\lambda)\,(A-\lambda B)^{-1}\,Bd\lambda \\ &= \frac{1}{2\pi i}\int_\gamma \left[(A-\lambda B)+(\lambda-\mu_0)B\right]f(\lambda)\,(A-\lambda B)^{-1}\,Bd\lambda \\ &= \frac{1}{2\pi i}\int_\gamma B\,(A-\lambda B)^{-1}\,B\,(\lambda-\mu_0)\,f(\lambda)\,d\lambda \\ &= \frac{1}{2\pi i}\int_\gamma \lambda B\,(A-\lambda B)^{-1}\,Bf(\lambda)\,d\lambda-\mu_0 T_l(f)\,B \\ &= \frac{1}{2\pi i}\int_\gamma B\,(A-\lambda B)^{-1}\,(\lambda B-A+A)\,f(\lambda)\,d\lambda-\mu_0 T_l(f)\,B \\ &= T_l(f)\,(A-\mu_0 B)\end{aligned}$$

Lemma is proved.

Definition 3. *Two operators $T_1 \in L(X)$, $T_2 \in L(Y)$ are called similar, if there is continuously invertible operator $V \in L(X;Y)$ such that $VT_2 = VT_1$. It is clear, that similar operators T_1 and T_2 have the same spectral properties and, particularly,*

$$\sigma(T_1) = \sigma(T_2)$$

Lemma 6. *Operators $R_r(\lambda_0)$ and $R_l(\lambda_0)$, $\forall\lambda_0 \in \rho(A,B)$ are similar.*

Proof. The equality

$$(A-\lambda_0 B)^{-1}\,R_l(\lambda_0) = R_r(\lambda_0)\,(A-\lambda_0 B)^{-1}\,,\ \lambda_0 \in \rho(A,B)\,,$$

takes place, such that the operators $R_l(\lambda_0)$ and $R_r(\lambda_0)$ are similar 7.

Corollary 7. *If the operator B is invertible, then*

$$\sigma\left(AB^{-1}\right) = \sigma\left(B^{-1}A\right) = \sigma(A,B)\,. \tag{2.8}$$

equality (2.8) follows from the equality $\sigma(R_l(\lambda_0)) = \sigma(R_r(\lambda_0))$. Thus, this corollary comes to the corollary 2 of lemma 1. Obtained result is applied for the investigation of spectral properties of quasi-similar operators.

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Definition 4. *Two operators $T_1 \in L(X)$, $T_2 \in L(Y)$ are called quasi-similar, if there is (not necessary continuously) invertible operator $V : X \to Y$ such that*

$$T_2 V = VT_1.$$

It is well known that two quasi-similar operators T_1 and T_2 can have non-coinciding spectrums. However from the corollary of lemma 6 we obtain that it takes place.

Theorem 8. *Let $T_1 \in L(X)$, $T_2 \in L(Y)$ be two quasi-similar operators, and $T_2B = BT_1$ for some invertible operator $B : X \to Y$. If the operator $T_2B - \lambda_0 B$ is continuously invertible for some $\lambda \in \mathbb{C}$. Then $\sigma(T_1) = \sigma(T_2)$.*

Now we'll do some limitations on the pairs of operators (A, B) from $L(X, Y)$:

1. *There is the number $\lambda_0 \in \mathbb{C}$, such that the operator $A - \lambda_0 B$ is continuously invertible, i.e. ρ regular set $\rho(A, B) \subset \mathbb{C}'$ is non-empty*
2. *Operators $R_r(\lambda; A, B)$, $\lambda \in \rho(A, B)$ from $L(X)$ and $R_l(\mu; A, B)$ from $L(Y)$, $\mu \in \rho(A, B)$ are mutually commutative.*

The least closed sub-algebras of the operators from $L(X)$ and $L(Y)$, containing all operators $R_r(\lambda; A, B)$, $\lambda \in \rho(A, B)$, $R_l(\mu; A, B)$, $\mu \in \rho(A, B)$ and identical operators I_X and I_Y correspondingly are denoted by B_r and B_l.

It is clear that B_r and B_l are commutative Banach algebras. Now we'll formulate one applied by us result on the spectral properties of pseudoresolvent ([2]*, theorem 5.8.4).*

Theorem 9. *Let $R : U \to L(X)$ is pseudoresolvent, defined on opened set U from C. Let B is the least complete (i.l. containing continuous inverse to each continuously invertible in $L(X)$ operator from B) subalgebra from $L(X)$, containing the operator I_X and all operators $R(\lambda)$, $\lambda \subset U$, and $\mathcal{M}$ is its space of maximal ideals.*

Then there is continuous function $\alpha : \mathcal{M} \to \tilde{\mathbb{C}}$ such that for all $\lambda \subset U$ it takes place the equality:

$$R(\lambda)(M) = (\lambda - \alpha(M))^{-1}, \ M \in \mathcal{M} \tag{2.9}$$

Further, $\sigma = \{\alpha(M) : M \in \mathcal{M}\}$ is closed subset of $\tilde{\mathbb{C}}$ and $\sigma \cap U = \emptyset$.

From Theorem 9 it follows that the function $\alpha : \mathcal{M} \to \bar{\mathbb{C}}$ constructs homeomorphism between $\mathcal{M}$ and closed subset σ from $\tilde{\mathbb{C}}$, and it allows to identify the space$\mathcal{M}$ with subset σ from $\tilde{\mathbb{C}}$.

We apply theorem 9 and now mention notation about pseudoresolvents R_l and R_r (without any limitations on the operator B) as it was done above, denoting by symbols B_l and B_r the least subalgebras from $L(Y)$ and $L(X)$ containing I_Y, $R_l(\lambda)$ and I_X, $R_r(\lambda)$, $\lambda \in \rho(A, B)$ correspondingly. The functions which are defined according to formula (2.9) are denoted by $\alpha_l : \mathcal{M}_l \to \tilde{\mathbb{C}}$ and $\alpha_r : \mathcal{M}_r \to \bar{\mathbb{C}}$ the functions, ($\mathcal{M}_l$ and $\mathcal{M}_r$ are the spaces of maximal ideals of algebras B_l and B_r correspondingly). If $\lambda_0 \in \rho(A, B)$, then $\sigma(R_l(\lambda_0)) = \sigma(R_r(\lambda_0))$ according to lemma 6 and because of it the sets $\alpha_l(\mathcal{M}_l)$ and $\alpha_r(\mathcal{M}_r)$ coincide, i.e.. the space of maximal ideals of algebras B_l and B_r are homeomorphic (coincide). So, from the obtained reasonings it follows [2].

Lemma 10. *Spaces of maximal ideals* $\mathcal{M}_l$ *and* $\mathcal{M}_r$ *of Banach algebras* B_l *and* B_r *are homeomorphic to closed subset* σ *from the extended complex plane* $\tilde{\mathbb{C}}$*, which has the property* $\sigma \cap \tilde{\rho}(A,B) = \emptyset$.

Definition 5. *Let the Banach spaces* X *and* Y *allow the decompositions* $X = X_1 \oplus X_2,\ Y = Y_1 \oplus Y_2$ *in direct sum of its closed subspaces* $X_i,\ Y_i,\ i = 1,2$. *We say that the pair* (A,B) *allows reducing relatively the painted decomposition of spaces, if* $AX_i \subset Y_i, \quad BX_i \subset Y_i,\ i = 1, 2$. *In this case we'll write*

$$(A,B) = (A_1 \oplus A_2,\ B_1 \oplus B_2) = (A_1,\ B_1) \oplus (A_2,\ B_2)\,, \tag{2.10}$$

where $A_i : X_i \to Y_i,\ i = 1,2$ *is narrowing of* A *on* X_i *and* $B_i : X_i \to Y_i,\ i = 1,2$ *is narrowing of* B *on* X_i. *Decomposition (2.10) is called the decomposition of pair* (A,B).

Obtained by this way pairs $(A_i,\ B_i)\,,\ i = 1,2$ *are called the reduced pairs relatively the pointed decomposition of spaces* $X,\ Y$.

Notation 1. *Let* $P_i \in L(X)\,,\ Q_i \in L(Y)\,,\ i = 1,2$ *be projectors, providing decompositions* $X = X_1 \oplus X_2,\ Y = Y_1 \oplus Y_2$, *i.l.* $P_iX = X_i,\ i = 1,2,\ Q_iY = Y_i,\ i = 1,2$. *Then it is called, that the pair* (A,B) *is reducing relatively the pointed decomposition of spaces if and only if the equalities take place:*

$$AP_i = Q_iA,\ BP_i = Q_iB,\ i = 1,2 \tag{2.11}$$

Notation 2. *If* $(A,\ B) = (A_1,\ B_1) \oplus (A_2,\ B_2)$, *then it takes place the equalities*

$$\begin{aligned} R(\lambda :\ A,\ B) &= (A - \lambda B)^{-1} = (A_1 - \lambda B_1)^{-1} \oplus (A_2 - \lambda B_2)^{-1}\,, \\ R_l(\lambda) &= B(A - \lambda B)^{-1} = B_1 (A_1 - \lambda B_1)^{-1} \oplus B_2 (A_2 - \lambda B_2)^{-1}\,, \\ R_r(\lambda) &= (A - \lambda B)^{-1} B = (A_1 - \lambda B_1)^{-1} B_1 \oplus (A_2 - \lambda B_2)^{-1} B_2. \end{aligned}$$

Now we'll prove the following theorem on absentness of the point ∞ *in spectrum* $\tilde{\sigma}(A,B)$ *of operators pair* $(A,\ B) \in L(X,\ Y)$.

Theorem 11. *Spectrum* $\tilde{\sigma}(A,B)$ *of the pair* (A,B) *of operators* $A,\ B : X \to Y$ *doesn't contain the point* ∞ *(particularly, it is bounded) if and only if the pair* (A,B) *allows the reducing of the form (2.10) relatively some decomposition*

$$X = X_1 \oplus X_2;\ Y = Y_1 \oplus Y_2 \tag{2.12}$$

and the following conditions are fulfilled: 1) operator $B_1 : X_1 \to Y_1$ *is continuously invertible and 2)* $A_2 : X_2 \to Y_2$ *is continuously invertible and* $\left(A_2^{-1} B_2\right)^2 = 0$.

Proof. Let $\infty \overline{\in} \sigma(A,B)$. Then the set of $\sigma(A,B)$ is compact. Consider the operator

$$C = \frac{1}{2\pi i} \int\limits_{\gamma} (A - \lambda B)^{-1}\, d\lambda,$$

where γ is Jordan contour of $\sigma(A,B)$. We'll show that the operators $CB = P_1 \in L(X)$ and $BC = Q \in L(Y)$ are projectors, satisfying the conditions

$$AP_1 = Q_1 A,\ BP_2 = Q_2 B \tag{2.13}$$

Consider the operator $P_1 = CB$. Then the operator

$$\begin{aligned} P_1 &= CB = \frac{1}{2\pi i}\int\limits_{\gamma} (A-\lambda B)^{-1} B d\lambda \\ &= \frac{1}{2\pi i}\int\limits_{\gamma} R_r(\lambda)\, d\lambda = T_r(1) \end{aligned}$$

is projector on the property of functional calculus for pseudoresolvent (functions $\lambda \to R_l(\lambda),\ \lambda \to R_r(\lambda)$ are halomorphic at the point ∞). Similarly it is proved that the operator

$$Q_1 = BC = T_l(1)$$

is projector.

We have to use the equality (2.6) from lemma 5, from which it follows the equality (2.12) and, consequently, the equality (2.11) (if we take into account that $P_1+P_2 = I_X,\ Q_1+Q_2 = I_Y$). Thus from the notation 1 it follows that the pair (A,B) allows the reducing of the form (2.10) relatively the decomposition of the form (2.12), where

$$X_i = P_i X,\ Y_i = Q_i Y,\ i = 1, 2.$$

From the pointed equalities $P_1 = CB,\ Q_1 = BC$ it follows that the operator $B_1 : X \to Y_1$ is continuously inverse, and $A_2^{-1}B_2$ is zero operator. Really, on Liouville theorem the operator $(A_2 - \lambda B_2)^{-1} B_2$ is constant operator $C_2 \in L(X)$. We obtain from here the following equalities

$$\begin{aligned} (A_2-\lambda B_2)^{-1} B_2 &= C_2 = \left(I_{x_2} - \lambda A_2^{-1} B_2\right)^{-1} A_2^{-1} B_2 \\ &= (I_{x_2} - \lambda A_0)^{-1} A_o, \end{aligned}$$

where $A_0 = A_2^{-1}B_2$. Because of this $C_2 = C_2 - \lambda C_2 A_0$ and $A_0 = C_2$. Consequently, $C_2 A_0 = A_0^2 = 0 = \left(A_2^{-1}B_2\right)^2 = 0$. Now let the pair (A,B) allow the reducing relatively some decomposition of the form (2.12), operators $B_1 : X_1 \to Y_1,\ A_2 : X_2 \to Y_2$ are continuously invertible and $\left(A_2^{-1}B_2\right)^2 = 0$. Then it is clear that

$$R_r(\lambda) = (A-\lambda B)^{-1} B = (A_1 - \lambda B_1) B_1 \oplus (A_2 - \lambda B_2)^{-1} B_2.$$

As $(A_1 - \lambda B_1)^{-1} B_1 = \left(A_1 B_1^{-1} - \lambda_{Y_1}\right)^{-1}$ is resolvent of bounded operator $A_1 B_1^{-1}$ then this function is halomorphic in ∞. Further

$$\begin{aligned} (A_2-\lambda B_2)^{-1} B_2 &= \left(I_{X_2} - \lambda A_2^{-1} B_2\right)^{-1} A_2^{-1} B_2 \\ &= \left(I_{X_2} + \lambda A_2^{-1} B_2\right) A_2^{-1} B_2 = A_2^{-1} B_2. \end{aligned}$$

Because of it the function $R_r(\lambda)$ is halomorphic at the point ∞, *i.e.* the point ∞ doesn't belong to $\tilde{\sigma}(A,B)$. ∎

Theorem is proved.

Corollary 12. $\sigma(A,B)=\sigma(A_1,B_1)$, *if* $\infty\overline{\in}\tilde{\sigma}(A,B)$.

Theorem 13. *Mentioned in lemma 10 set σ coincides with extended spectrum $\tilde{\sigma}(A,B)$ of pair (A,B), and for any $\lambda_0\in\rho(A,B)$ the equalities*

$$\sigma(R_l(\lambda_0))=\sigma(R_r(\lambda_0))=\left\{\frac{1}{\lambda-\lambda_0}:\lambda\in\tilde{\sigma}(A,B)\right\}$$

take place.

Proof. If $\infty\overline{\in}\tilde{\sigma}(A,B)$, then from previous theorem we obtain that (see also its corollary) without limitation of generality we can consider the operator B continuously invertible (otherwise we have to consider the reducing pair with invertible operator B_1). However in the case of invertible B, theorem 13 is obvious. Now we consider the case, when $\infty\overline{\in}\tilde{\sigma}(A,B)$. Let λ_0 be some point from $\rho(A,B)\neq\emptyset$. From the equality

$$(A-\lambda B)=(A-\lambda_0 B)(I_X-(\lambda-\lambda_0)R_r(\lambda_0))$$

it follows that $\lambda_0\neq\lambda\in\rho(A,B)$ if and only if the number $(\lambda-\lambda_0)^{-1}$ doesn't belong to spectrum of operator $R_r(\lambda_0)$.

From the formula (1.1) and lemma 10 we obtain that $\lambda\overline{\in}\sigma$ if and only if $\lambda\in\rho(A,B)$. On the condition $\infty\in\tilde{\sigma}(A,B)$, and according to theorem 13 the operator $R_r(\lambda_0)$ is not invertible, *i.e.* $\infty\in\sigma$. Similar reasonings are done for $R_l(\lambda_0)$. Theorem is proved. ∎

Corollary 14. *If B is finite-dimensional operator, then $\sigma(A,B)$ is finite set, the number of elements of which is not more that the rang of operator B (dimension of the set of values of operator B). If X is infinite-dimensional space and co-dimension of the values set of the operator A is equal to ∞, then $\infty\overline{\in}\sigma(A,B)$.*

Proof. Let $\lambda_0\in\rho(A,B)$. Then the set $\tilde{\sigma}(A,B)$ homeomorphic $\sigma(R_l(\lambda_0))$. As the operator $R_l(\lambda_0)=B(A-\lambda_0 B)^{-1}$ is finite dimensional, the number of the points in $\sigma(A,B)$ is not more than the rang of operator $R_l(\lambda_0)$ and, consequently, the rang of operator B.

The second part of corollary is direct corollary of theorem 13. Corollary is proved. From the reasonings, similar processes are done for the proof of corollary 7, we obtain once more. ∎

Corollary 15. *If B is absolutely continuous operator, then $\tilde{\sigma}(A,B)$ is not more than countable set with unique possible limit point, which equals to ∞.*

Theorem 16. *For any function f from $H(\Delta)$ the equalities*

$$\sigma(T_r(f))=\sigma(T_l(f))=f(\tilde{\sigma}(A,B))=\{f(\lambda):\lambda\in\tilde{\sigma}(A,B)\}$$

take place.

Proof. Consider the Banach algebra $B_r\left(\subset L(X)\right)$, introduced by us earlier. Then from the formula (2.8) for any maximal ideal $m \in \mathcal{M}_r = \tilde{\sigma}(A,B) = \sigma$ it takes place the equality

$$\begin{aligned} T_r(f)(M) &= \delta f(\infty) + \frac{1}{2\pi i}\int\limits_{\gamma} f(\lambda)\,.\frac{1}{(\lambda - \alpha_r(M))}d\lambda \\ &= f(\alpha_r(M))\,. \end{aligned}$$

As the operator $T_r(f)$ belongs to algebra B_r,

$$\sigma(T_r(f)) = \{T_r(f)(M) : M \in \mathcal{M}_r = \sigma(A,B)\} = f(\tilde{\sigma}(A,B))\,.$$

The equality $\sigma(T_l(f)) = f(\tilde{\sigma}(A,B))$ is proved similarly. Theorem is proved. ∎

3. **Application**

In this part we consider the linear differential equation of the second order of the following type:

$$A\ddot{X} + B\dot{X} + CX = \psi(t)\,, \tag{3.1}$$

where $A : D(A) \subset X \to X,\ B : D(B) \subset X \to X,\ C : D(C) \subset X \to X$ are linear options, acting in complex Banach space X and having the definition domains $D(A),\ D(B),\ D(C)$ resp. such that the supspace $X_0 = D(A) \cap D(B) \cap D(C)$ is dense in X . We make various assumptions about the function $\psi : (a,f) \to X$. Nevertheless one can obtain the most important results in that case, where ψ belongs to the Banach space $C(R;X)$ of the X valued continuous on RR functions and especially to the subspace of continuous almost periodical functions [3]

The case $A,B,C \in L(X;Y)$ and $\psi \in B(R;Y)$is considered separately. The equation (3.2) for $\psi \in C(R;X)$ is conveniently to consider as the operator equation of the following type

$$L_X = \psi,$$

where $L : D(L) \subset C(R;X) \to C(R;X)$ is linear operator

$$L_X \equiv A\ddot{X} + B\dot{X} + CX,$$

with appropriate definition domain $D(L)$.

The problem to determine the definition domain $D(L)$ is very complex in the case when at least on of A,B,C is non-bounded operator.

To determine $D(L)$ one ought to make some additional assumption on the operators A,B,C considering the operator-valued function H defined on R, with values in the set of linear closed operators. This function is determined by the formula $H(\lambda) = \lambda^2 A + iB\lambda + C$ under the assumption that each of operators $H(\lambda) : D(H(\lambda)) \subset X \to X$ is closed extension with $X_0 = D(A) \cap D(B) \cap D(C)$ and $D(H(\lambda)) \supset D(A),\ D(H(\lambda)) \supset D(B)$. Let us make the main assumptions respected to the bundle of operators $H(\lambda),\ \lambda \in$

C: There exists such complex number Z_0, that the operator-valued function $H_0(\lambda) = H(\lambda) + Z_0 I$ satisfies the conditions:

Each of operators $H_0(\lambda),\ \lambda \in R$ has continuous inverse one and $H_0^{-1}(\lambda)$ admits the estimation of the following type:

$$\left\|H_0^{-1}(\lambda)\right\| \le \frac{const}{(1+|\lambda|)^{1+\alpha}},\ \forall\lambda \in R \tag{3.2}$$

where $\alpha > 0$ is some number and const is absolute constant, independent on $\lambda \in R$;

Operators $AH_0^{-1}(\lambda),\ BH_0^{-1}(\lambda)$ are bounded and

$$\left\|AH_0^{-1}(\lambda)\right\| \le \frac{const}{(1+|\lambda|)},\ \left\|BH_0^{-1}(\lambda)\right\| \le const \tag{3.3}$$

for each $\lambda \in R$ (const-the absolute constant, independent on $\lambda \in R$) the trying of the conditions (3.2)-(3.3) is simple than the trying the correctness condition of the Cauchy problem for respective homogeneous equation (3.2). ($\psi = 0$).

Note : In the case $A, B, C \in L(X;Y)$ the respective function $H : R \to L(X;Y)$ is defined

$$H(\lambda) = (i\lambda)^2 A + (i\lambda) B + C \ \ (*)$$

and about this function one ought to make the following assumption, different from above ones:

Suppose that the space X is continuously imbedded into Y and there exists $Z_0 \in C$ such that function $H_0(\lambda) = H(\lambda) + Z_0 I_*$, where $I_* : X \to Y$ the imbedding operator has continuous inverse for each $\lambda \in R$ and analogs of the conditions (3.2)-(3.3) holds

$\left\|H_0^{-1}(\lambda)\right\| = \frac{const}{(1+|\lambda|)^{1+\alpha}},\ \alpha > 0\,(3.2)'$

$\left\|AH_0^{-1}(\lambda)\right\| = \frac{const}{1+|\lambda|},\ \left\|BH_0^{-1}(\lambda)\right\| \le const, (3.3)'$

where $\lambda \in R$ and const is absolute constant, $H_0^{-1} : R \to L(Y,X)$,

$AH_0^{-1}(\lambda),\ BH_0^{-1}(\lambda) \in L(Y),\quad \lambda \in R.$

Lemma 17. *The function $H_0(\lambda)$ satisfying the continuous (3.2)-(3.3) (resp. $H_0 : R \to L(X;Y)$) satisfying conditions (3.2)-(3.3) has the property: the exists the continuous operator-valued summable function $G : R \to L(X)$ (resp. $G : R \to L(Y;X)$) such that*

$$H_0^{-1}(\lambda) = \int_{-\infty}^{\infty} G(t)\, e^{-i\lambda t} dt,\ \lambda \in R$$

i.e. the function $H_0^{-1}(\lambda)$ is the Fourier transformation of the summable function $G(t)$ ($\int_{-\infty}^{\infty} \|G(t)\|\, dt = +\infty$).

Proof. Set

$$G(t) = \int_{-\infty}^{\infty} H_0^{-1}(\lambda)\, e^{i\lambda t} d\lambda,\ t \in R, \tag{3.4}$$

From the condition of summability of the function $H_0^{-1}(\lambda)$ (which follows from the condition (3.2) or (3.2)′) follows that this function is continuous and bounded. (Moreover, $\|G(t)\| \to 0$ as $|t| \to \infty$). From the definition of the function $H_0^{-1}(\lambda)$ and conditions (3.3) or (3.3)′ immediately follows that

$$\frac{dH_0^{-1}(\lambda)}{d\lambda} = H_0^{-1}(\lambda)\left(-2A\lambda + iB\right) H_0^{-1}(\lambda)$$

and

$$\left\| \frac{dH_0^{-1}(\lambda)}{d\lambda} \right\| \le const \left\| H_0^{-1}(\lambda) \right\| \left(\frac{|\lambda|}{1+|\lambda|} + 1 \right) \le \frac{const}{1+|\lambda|},\ \lambda \in R,$$

i.e. the function $H_0^{-1}(\lambda)$ is continuously differentiable amd summable on R.One can analogously prove that it has the second derivative $\frac{d^2 H_0^{-1}}{d\lambda^2}$ and

$$\left\| \frac{d^2 H_0^{-1}(\lambda)}{d\lambda^2} \right\| \le \frac{const}{(1+|\lambda|)^{\alpha}}.$$

Taking it into account from (3.4) one can obtain the inequality

$$-t^2 G(t) = \int_{-\infty}^{\infty} G(t)\, x_0 e^{i\lambda_o t} dt$$

belongs to $D(L)$ and therefore from this equality follows that the function $y_0 e^{i\lambda t}$ belongs to $D(L)$ if $y_0 \in D(H_0(\lambda_0))$. For $D(H_0(\lambda_0)) = D(H(\lambda_0))$, the function $y_0 e^{i\lambda_0 t}$ belongs to $D(L)$ if $y_0 \in D(H(\lambda_0))$. ∎

Definition 6. *Continuous bounded function $\varphi : R \to X$ is called the generalized bounded solution of the equation (3.2) if it satisfies the equation*

$$\varphi(t) = \int_{-\infty}^{\infty} G(t-s)\left(\psi(s) + z_0 \varphi(s)\right) ds$$

This approach to the definition of generalized solution of the equation (3.2) is very convinied from operator viewpoints. The main reason of it is that the conditions (3.2)-(3.3) on bundle make possible to define the natural definition domain $D(L)$ of the operator $L : D(L) \subset C(R;X) \to C(R;X)$ in the Banach space $C(R;X)$ of the bounded on R functions with values on X, defined by the differential expression:

$$LX \equiv A\ddot{X} + B\dot{X} + CX.$$

In order to define $D(L)$ we produced as follows: firstly we define the definition domain of the operator $L + Z_0 I$ and then write : $D(L) = D(L + Z_0 I)$.

Definition 7. *The Beurling spectrum of a continuous bounded function φ : $R \to X$ is defined to be the set of common zeros of the Fourier transforms of the functions in the set*

$$\{g \in L_1(R) : g * \varphi = 0\},$$

where $L_1(R)$ is the Banach space of integrable complex-valued functions on R with convolution of functions as multiplication. The Beurling spectrum of φ is denoted by $S(\varphi)$.

It is not hard to show that the Beurling spectrum of φ coincides with the support of its Fourier transform (if it is regarded as a generalization function of slow grouth).

Definition 8. *The set of non-almost-periodic of a continuous bounded function $\varphi : R \to X$ is defined to be the common set of zeros of the Fourier transforms of the functions in the set*

$$\{g \in L_1(R) : g * \varphi \quad \text{is an}\, H\, P \text{ function}\}$$

The set of non-almost periodicity of a function φ is denoted by $S_0(\varphi)$.

Definition 9. *The singular set of the pencil $M(\lambda)$, $\lambda \in R$, defined by (*) is defined to be the complement in R of the set $\{\lambda \in R\}$: the operator $M(\lambda)$ has a continuous inverse.*

The singular set is closed and is denoted by $S(M)$.

Theorem 18. *The set $S_0(\varphi)$ of non-almost-periodicity of a generalized solution $\varphi : R \to X$ of equation (3.2) is contained in the singular set of $S(M)$ of the pencil $M(\lambda)$, $\lambda \in R$. In particular, φ is an HP function if $S(M) = \emptyset$. Moreover, $S_0(\varphi) \subset S(M) \cup S(\varphi)$.*

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