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Rough Set Theory Applied to BCC-Algebras

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Abstract

As a generalization of subalgebras/ideals in BCC-algebras, the notion of rough subalgebras/ideals is introduced, and some of their properties are discussed.

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1 Introduction

In 1982, Pawlak introduced the concept of a rough set (see [6]). This concept is fundamental for the examination of granularity in knowledge. It is a concept which has many applications in data analysis (see [7]). Rough set theory is applied to semigroups and groups (see [4, 5]), and BCK-algebras (see [3]). In this paper, we apply the rough set theory to BCC-algebras, and we discuss the notion of upper/lower rough subalgebras/ideals which is an extended notion of a subalgebra/ideal in a BCC-algebra.

2 Preliminary Notes

Recall that a *BCC-algebra* is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms:

(C1) $((x * y) * (z * y)) * (x * z) = 0$,

(C2) $0 * x = 0$,

(C3) $x * 0 = x$,

(C4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

for every x, y, $z \in X$. For any BCC-algebra X, the relation $\leq$ defined by $x \leq y$ if and only if $x * y = 0$ is a partial order on X. In a BCC-algebra X, the following holds (see [2]).

(p1) $x \leq x$,

(p2) $x * y \leq x$,

(p3) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$

for all x, $y \in X$. A nonempty subset S of a BCC-algebra X is said to be a *subalgebra* of X if $x * y \in S$ whenever x, $y \in S$. A nonempty subset A of a BCC-algebra X is called an *ideal* of X, denoted by $A \trianglelefteq X$, if it satisfies

(I1) $0 \in A$,

(I2) $(x * y) * z \in A$ and $y \in A$ imply $x * z \in A$ for all x, y, $z \in X$.

Note that an ideal of a BCC-algebra X is a subalgebra of X.

Let V be a set and ρ an equivalence relation on V and let $\mathcal{P}(V)$ denote the power set of V. For all $x \in V$, let x/ρ denote the equivalence class of x with respect to ρ. Define the functions $\rho_-, \rho^- : \mathcal{P}(V) \to \mathcal{P}(V)$ as follows: $\forall S \in \mathcal{P}(V)$,

$$\rho_-(S) = \{x \in V \mid x/\rho \subseteq S\} \quad \text{and} \quad \rho^-(S) = \{x \in V \mid x/\rho \cap S \neq \emptyset\}.$$

The pair (V, ρ) is called an *approximation space*. Let S be a subset of V. Then S is said to be *definable* if $\rho_-(S) = \rho^-(S)$ and *rough* otherwise. $\rho_-(S)$ is called the *lower approximation* of S while $\rho^-(S)$ is called the *upper approximation*.

3 Rough subalgebras/ideals

In what follows, let X denote a BCC-algebra unless otherwise specified. Let ρ be a congruence relation on X, that is, ρ is an equivalence relation on X such that $(x, y) \in \rho$ implies $(x * z, y * z) \in \rho$ and $(z * x, z * y) \in \rho$ for all $z \in X$. We denote by X/ρ the set of all equivalence classes of X with respect to ρ, that is, $X/\rho := \{x/\rho \mid x \in X\}$.

Lemma 3.1 ([1]). *If ρ is a congruence relation on X, then $0/\rho$ is an ideal of X.*

Define an operation $*$ on X/ρ by $x/\rho * y/\rho = (x*y)/\rho$ for all x/ρ, $y/\rho \in X/\rho$. Observe that $(X/\rho, *, 0/\rho)$ may not be a BCC-algebra, because X/ρ does not satisfy the condition (C4) (see [1]).

Theorem 3.2. *The following are equivalent:*

(1) $\forall x, y \in X$, $x * y \in 0/\rho$ *and* $y * x \in 0/\rho$ *imply* $(x, y) \in \rho$.

(2) ρ *is regular, that is,* $x/\rho * y/\rho = 0/\rho = y/\rho * x/\rho$ *implies* $x/\rho = y/\rho$.

(3) $(X/\rho, *, 0/\rho)$ *is a BCC-algebra.*

Proof. (1) $\Rightarrow$ (2) Suppose $x/\rho * y/\rho = 0/\rho = y/\rho * x/\rho$. Then $(x * y)/\rho = 0/\rho = (y * x)/\rho$, and so $(x * y, 0) \in \rho$ and $(y * x, 0) \in \rho$. It follows from (1) that $(x, y) \in \rho$, that is, $x/\rho = y/\rho$.

(2) $\Rightarrow$ (3) Straightforward.

(3) $\Rightarrow$ (1) Let $x, y \in X$ be such that $x * y \in 0/\rho$ and $y * x \in 0/\rho$. Then $x/\rho * y/\rho = (x * y)/\rho = 0/\rho = (y * x)/\rho = y/\rho * x/\rho$. It follows from (C4) that $x/\rho = y/\rho$, that is, $(x, y) \in \rho$. This completes the proof. □

□

For any nonempty subsets A and B of X, we define

$$A * B := \{x * y \mid x \in A, y \in B\}.$$

Proposition 3.3. *Let ρ be a congruence relation on X. For any nonempty subsets A and B of X, we have*

(1) $\rho_-(A) \subseteq A \subseteq \rho^-(A)$,

(2) $\rho^-(A \cup B) = \rho^-(A) \cup \rho^-(B)$,

(3) $\rho_-(A \cap B) = \rho_-(A) \cap \rho_-(B)$,

(4) $A \subseteq B$ *implies* $\rho_-(A) \subseteq \rho_-(B)$ *and* $\rho^-(A) \subseteq \rho^-(B)$,

(5) $\rho_-(A \cup B) \supseteq \rho_-(A) \cup \rho_-(B)$,

(6) $\rho^-(A \cap B) \subseteq \rho^-(A) \cap \rho^-(B)$,

(7) $\rho^-(A) * \rho^-(B) \subseteq \rho^-(A * B)$,

(8) $\rho_-(A) * \rho_-(B) \subseteq \rho_-(A * B)$ *whenever*

$$\rho_-(A) * \rho_-(B) \neq \emptyset \text{ and } \rho_-(A * B) \neq \emptyset.$$

Proof. It is straightforward. □

Definition 3.4. A nonempty subset S of X is called an *upper* (resp. a *lower*) *rough subalgebra* (or, *ideal*) of X if the upper (resp. nonempty lower) approximation of S is a subalgebra (or, ideal) of X. If S is both an upper and a lower rough subalgebra (or, ideal) of X, we say that S is a *rough subalgebra* (or *ideal*) of X.

Theorem 3.5. *Every subalgebra of X is a lower rough subalgebra of X.*

Proof. Let S be a subalgebra of X. Taking $A = B = S$ in Proposition 3.3(8), we have

$$\rho_-(S) * \rho_-(S) \subseteq \rho_-(S * S) \subseteq \rho_-(S)$$

because S is a subalgebra of X. Hence $\rho_-(S)$ is a subalgebra of X, that is, S is a lower rough subalgebra of X. □

Theorem 3.6. *Every subalgebra of X is an upper rough subalgebra of X.*

Proof. Let S be a subalgebra of X. We show that $\rho^-(S)$ is a subalgebra of X. Let $x, y \in \rho^-(S)$. Then $x/\rho \cap S \neq \emptyset$ and $y/\rho \cap S \neq \emptyset$. Thus there exist a_x, $a_y \in S$ such that $a_x \in x/\rho$ and $a_y \in y/\rho$. It follows that $(a_x, x) \in \rho$ and $(a_y, y) \in \rho$ so that $(a_x * a_y, x * y) \in \rho$, that is, $a_x * a_y \in (x * y)/\rho$. On the other hand, since S is a subalgebra of X, we have $a_x * a_y \in S$. Hence $a_x * a_y \in (x*y)/\rho \cap S$, that is, $(x*y)/\rho \cap S \neq \emptyset$. This shows that $x*y \in \rho^-(S)$. Therefore S is an upper rough subalgebra of X. □

For any $A \trianglelefteq X$, define a relation ρ on X by

$$(x, y) \in \rho \text{ if and only if } x * y \in A \text{ and } y * x \in A.$$

Then ρ is an equivalence relation on X related to A. Moreover ρ is a regular congruence relation on X for any $A \trianglelefteq X$, and $0/\rho = A$ and $X/\rho := X/A$ is a BCC-algebra. (see [1]).

The following example shows that the converse of Theorem 3.6 may not be true. Hence we know that the notion of an upper rough subalgebra is an extended notion of a subalgebra.

Example 3.7. (1) Let $X = \{0, a, b, c\}$ be a BCC-algebra with the Cayley table as follows.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Consider $A = \{0, a\} \trianglelefteq X$ and let ρ be an equivalence relation on X related to A. Then $0/\rho = a/\rho = A$, $2/\rho = \{2\}$, and $3/\rho = \{3\}$. Consider a subset $S = \{a, b\}$ of X which is not a subalgebra of X. Then

$$\rho^-(S) = \{x \in X \mid x/\rho \cap S \neq \emptyset\} = \{0, a, b\}$$

which is a subalgebra of X.

(2) Let $X = \{0, 1, 2, 3, 4\}$ be a proper BCC-algebra with the Cayley table as follows.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Note that $A = \{0, 1, 2, 3\}$ is an ideal of X. Let τ be an equivalence relation on X related to A. Then $0/\tau = 2/\tau = A$, $1/\tau = 3/\tau = X$, and $4/\tau = \{1, 3, 4\}$. Moreover,

$$\tau^-(\{4\}) = \{x \in X \mid 4 \in x/\tau\} = \{1, 3, 4\}$$

which is not a subalgebra of X and

$$\tau^-(\{2\}) = \{x \in X \mid 2 \in x/\tau\} = \{0, 1, 2, 3\}$$

which is a subalgebra of X.

Proposition 3.8. *If ρ is an equivalence relation on X related to the trivial ideal $\{0\}$, then every subset of X is definable.*

Proof. Let S be a subset of X. Then

$$\begin{aligned} x/\rho &= \{y \in X \mid (x, y) \in \rho\} \\ &= \{y \in X \mid x * y = 0,\ \ y * x = 0\} \\ &= \{y \in X \mid x = y\} = \{x\} \end{aligned}$$

for all $x \in X$. It follows that

$$\rho_-(S) = \{x \in X \mid x/\rho \subseteq S\} = \{x \in X \mid \{x\} \subseteq S\} = S$$

and

$$\rho^-(S) = \{x \in X \mid x/\rho \cap S \neq \emptyset\} = \{x \in X \mid \{x\} \cap S \neq \emptyset\} = S.$$

Hence S is definable. □

Proposition 3.9. *Every BCC-algebra is definable with respect to any ideal.*

Proof. Straightforward. □

Proposition 3.10. *If ρ and τ are equivalence relations on X related to ideals A and B of X, respectively, then $\rho \subseteq \tau$ whenever $A \subseteq B$.*

Proof. If $(x, y) \in \rho$, then $x * y \in A \subseteq B$ and $y * x \in A \subseteq B$. Hence $(x, y) \in \tau$, and so $\rho \subseteq \tau$. □

Let ρ be an equivalence relation on X related to an ideal A of X. If B is an ideal of X such that $A \neq B$, then $\rho_-(B)$ is not an ideal of X in general as seen in the following example.

Example 3.11. Let $X = \{0, a, b, c, d\}$ be a BCC-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	b	0
c	c	c	c	0	0
d	d	d	d	d	0

Consider $A = \{0, a, b\} \trianglelefteq X$ and let ρ be an equivalence relation on X related to A. If we take $B = \{0, a, c\} \trianglelefteq X$, then $A \neq B$ and $\rho_-(B) = \{3\}$ which is not an ideal of X.

Example 3.7(2) shows that there exists a non ideal U of X for which U is an upper rough ideal of X with respect to an ideal of X. Hence we know that the notion of an upper rough ideal is an extended notion of an ideal.

Theorem 3.12. *Let ρ be an equivalence relation on X related to an ideal A of X. If U is an ideal of X containing A, then $\rho_-(U)$ $(\neq \emptyset)$ is an ideal of X, that is, U is a lower rough ideal of X.*

Proof. Let U be an ideal of X containing A. Let $x \in 0/\rho$. Then $x = x * 0 \in A \subseteq U$, and so $0/\rho \subseteq U$. Hence $0 \in \rho_-(U)$. Let x, y, $z \in X$ be such that $(x * y) * z \in \rho_-(U)$ and $y \in \rho_-(U)$. Then $y/\rho \subseteq U$ and

$$(x/\rho * y/\rho) * z/\rho = ((x * y) * z)/\rho \subseteq U.$$

Let $w \in (x*z)/\rho = x/\rho*z/\rho$. Then $w = a_x*a_z$ for some $a_x \in x/\rho$ and $a_z \in z/\rho$. From $a_x \in x/\rho$ and $a_z \in z/\rho$, we have $(a_x,\, x) \in \rho$ and $(a_z,\, z) \in \rho$. Taking $a_y \in y/\rho$, then $(a_y,\, y) \in \rho$. Since ρ is congruence, we get $\big((a_x*a_y)*a_z, (x*y)*z\big) \in \rho$ and so $(a_x * a_y) * a_z \in ((x * y) * z)/\rho \subseteq U$. Since U is an ideal of X, it follows from (I2) that $w = a_x * a_z \in U$ so that $(x * z)/\rho \subseteq U$. Hence $x * z \in \rho_-(U)$, and therefore $\rho_-(U)$ is an ideal of X. □

Before finishing our discussion, we pose an open problem: Let ρ and U be as in Theorem 3.12. Is U an upper rough ideal of X?

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