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Injectivity of the Determinantal Map for Space of Sections of Stable Vector Bundles on Curves

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Abstract. Let X be a smooth aprojective curve of genus $g \geq 2$ and E a rank n vector bundle on X. For any linear subspace W of E let $\tau_{E,W} : \bigwedge^n(W) \to H^0(X, \det(E))$ denote the exterior product map. Here we study the existence of triple (X, E, V) such that E is stable and $\tau_{E,W}$ is injective, essentially when $g \gg n$.

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Let X be a smooth and connected projective curve of genus $g \geq 2$ and E a rank n vector bundle on X. For any linear subspace W of E let $\tau_{E,W} : \bigwedge^n(W) \to H^0(X, \det(E))$ denote the exterior product map. The study of injectivity of $\tau_{E,W}$ was started by N. Teixidor i Bigas in [4]. She also linked it the Plüker embedding of the Grassmannian. Here (stimulated by [3]. §4) we will consider it again and prove the following result.

Theorem 1. *Fix integers n, b, k, g, d such that $b > n \geq 2$, $g \geq 2k - 1 \geq 3$, set $u := \lceil \binom{b}{n}/n \rceil$. Assume $d \geq g + ukn + 1$. Let X be a smooth and connected k-gonal curve of genus g. Fix $L \in Pic^d(X)$. Then there exist a rank n stable vector bundle E and a b-dimensional linear subspace V of $H^0(X, E)$ such that $\det(E) \cong L$ and the exterior product map $\tau_{E,V} : \bigwedge^n(V) \to H^0(X, \det(E))$ is injective.*

Theorem 2. *Fix integers n, b, k, g, d such that $b > n \geq 2$, $g \geq 2k - 1 \geq 3$, set $u := \lceil \binom{b}{n}/n \rceil$. Assume $d \geq ukn + n + 1$. Let X be a smooth and connected*

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k-gonal curve of genus g. There is an integral family $U \subseteq Pic^d(X)$ such that $\dim(A) = \min\{g, d - ukn\}$ *and for all $L \in A$ the following is true. Then there exist a rank n stable vector bundle E and a b-dimensional linear subspace V of $H^0(X,E)$ such that* $\det(E) \cong L$ *and the exterior product map $\tau_{E,V} : \bigwedge^n(V) \to H^0(X, \det(E))$ is injective.*

Remark 1. The case $b = 2n+1$, $d = 2g-2$ of Theorem 1 give a partial extension (E is not spanned in Theorem 1) to the case $n \geq 3$ of [3], part (ii) of Theorem 4.1. However, here we also get the existence part for stable vector bundles, while in [3] only semistable and decomposable solutions were given. We are also more explicit on the choice of the curve X.

Remark 2. Here we will follow [2], §2. For all integers $b > a > 0$ let $G(a,b)$ denote the Grassmannian of all $(b-a)$-dimensional linear subspaces of $\mathbb{K}^{\oplus b}$. Thus $\dim(G(a,b)) = a(b-a)$, $\mathrm{Pic}(G(a,b)) \cong \mathbb{Z}$ and the positive generator $\mathcal{O}_{G(a,b)}(1)$ of $\mathrm{Pic}(G(a,b))$ induces the Plücker embedding $j_{a,b}$ of $G(a,b)$ into $\mathbf{P}^{N(a,b)}$, $N(a,b) := \binom{b}{a} - 1$. To simplify the notations we will identify $G(a,b)$ with $j_{a,b}(G(a,b))$. Let

$$0 \to S_{a,b} \to \mathcal{O}_{G(a,b)}^{\oplus n} \to Q_{a,b} \to 0 \tag{1}$$

be the tautological exact sequence of $G(a,b)$. Thus $Q_{a,b}$ is the rank a tautological quotient bundle of $G(a,b)$, $S_{a,b}$ is the tautological rank $(b-a)$ subbundle of $G(a,b)$, $\det(Q_{a,b}) \cong \det(S_{a,b})^* \cong \mathcal{O}_{G(a,b)}(1)$ and $TG(a,b) \cong Q_{a,b} \otimes S_{a,b}^*$. Set $V_{a,b} := H^0(G(a,b), Q_{a,b})$. Hence $\dim(V_{a,b}) = b$ and the determinantal map $\tau_{a,b} : \bigwedge^a(V_{a,b}) \to H^0(G_{a,b}, \mathcal{O}_{G(a,b)}(1))$ is bijective. For every closed subscheme $T \subset G(a,b)$ let $V_{T,a,b}$ denote the image og the restriction map $H^0(G(a,b), Q_{a,b}) \to H^0(T, Q_{a,b}|T)$. A curve X in a projective space $\mathbf{P}^N$ is called a chain of d lines if it is reduced, connected, nodal, it has exactly d irreducible components $C_1, \dots, C_d$, each of them being a line, and $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Hence every chains of lines has arithmetic genus 0 and it is smoothable inside its ambient projective space. Fix any chain $X = C_1 \cup \cdots \cup C_d \subset G(a,b) \subset \mathbf{P}^{N(a,b)}$ of d lines. X is smoothable inside $G(a,b)$, it is a smooth point of the Hilbert scheme $\mathrm{Hilb}(G(a,b))$ of $G(a,b)$ and $h^1(X, N_{X,G(a,b)}) = 0$ ([2], Remark 2.5) For all i the vector bundle $Q_{a,b}|C_i$ is a direct sum of a line bundle of degree 1 and $a-1$ line bundles of degree 0. Using $d-1$ Mayer-Vietoris exact sequences we easily get $h^0(X, Q_{a,b}|X) = d+a$ and $h^1(X, Q_{a,b}|X) = 0$. For all integers a, b, x such that $1 \leq x \leq N(a,b)$ there is a chain $Y \subset G(a,b)$ of x lines whose spans in $\mathbf{P}^{N(a,b)}$ has dimension x ([2], Lemma 2.7).

Example 1. Fix integers $b > a > 0$ and set $u := \lceil N(a,b)/a \rceil$ and $\epsilon := ua - N(a,b)$. Hence $0 \leq \epsilon < a$. By Remark 2 there is a chain $T \subset G_{a,b}$ of $N(a,b)$ lines spanning $\mathbf{P}^{N(a,b)}$. Hence $\dim(V_{T,a,b}) = b$ and the determinantal map $\tau_{T,a,b} : \bigwedge^a(V_{T,a,b}) \to H^0(T, \mathcal{O}_T(1))$ is bijective. By Remark 2 T is smoothable inside $G(a,b)$. Take a general such smoothing $C \subset G(a,b)$. Hence $C \cong \mathbf{P}^1$,

C spans $\mathbf{P}^{N(a,b)}$, and $\deg(C) = N(a,b)$. By semicontinuity $\dim(V_{C,a,b}) = b$ and the determinantal map $\tau_{C,a,b} : \bigwedge^a(V_{C,a,b}) \to H^0(C, \mathcal{O}_C(1))$ is bijective. Identify C with an abstract $\mathbf{P}^1$ and let F be the rank a vector bundle on $\mathbf{P}^1$ which is the direct sum of ϵ line bundles of degree $u-1$ and $a-\epsilon$ line bundles of degree u. Thus F is the only rigid vector bundle on $\mathbf{P}^1$ such that $\deg(F) = \deg(Q_{a,b}|C)$ and $\mathrm{rank}(F) = \mathrm{rank}(Q_{a,b}|C)$. Since $Q_{a,b}$ is spanned and C is smooth and rational, $h^1(C, Q_{a,b}|C) = 0$. Thus $b = h^0(C, Q_{a,b}) = h^0(\mathbf{P}^1, F)$. $Q_{a,b}|C$ is the flat limit of an integral family of vector bundles on $\mathbf{P}^1$ isomorphic to F. By semicontinuity we get the bijectivity of the determinantal map $\bigwedge^a(H^0(\mathbf{P}^1, F)) \to H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(N(a,b))$ is bijective.

Proof of Theorem 1. Set $\epsilon := un - N(n,b)$. Let F be the vector bundle on $\mathbf{P}^1$ considered in Example 1. Fix a degree k morphism $f : X \to \mathbf{P}^1$. Set $G := f^*(F)$ and $V := f^*(H^0(\mathbf{P}^1, F)) \subseteq H^0(X, G)$. Hence $\mathrm{rank}(G) = n$, $\deg(G) = ukn - k\epsilon$, V spans G and $\dim(V) = b$. By Example 1 the determinant map $\tau : \bigwedge^n(V) \to H^0(G, \det(G))$ is injective. Since $d \geq g + ukn + 1$, there is an effective and reduced degree $d - \deg(G)$ divisor D such that $\det(G)(D) \cong L$. Let E be the general vector bundle obtained from F making $d - \deg(G)$ general positive elementary transformations, each of them supported by a different point of the support of D. The proof of [1], Cor. 2.4 and Prop. 2.7, gives the stability of E. Since G is a subsheaf of G, we may see V as a linear subspace of $H^0(X, E)$. Since $\tau_{G,V}$ is injective, $\tau_{E,V}$ is injective. □

Proof of Theorem 2. The family A is the family $\{\det(G)(D)\}$ in which D varies in a non-empty open subset of the symmetric product of $d - ukn$ copies of X. □

We work over an algebraically closed field $\mathbb{K}$. For the statement of Theorem 1 we require $\mathrm{char}(\mathbb{K}) = 0$.

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