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Rank 2 Vector Bundles with Canonical Determinant on a Smooth Curve

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Abstract. We prove an existence theorem for rank 2 stable vector bundles with many sections, canonical determina and low degree of stability on a general smooth projective curve.

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Let X be a smooth projective curve of genus $g \geq 3$. Let $B(X)$ denote the set of all rank 2 stable vector bundles E on X such that $\det(E) \cong \omega_X$. For every integer $n > 0$ set $B(X,n) := \{E \in B(X) : h^0(X,E) = n\}$. Concerning $B(X,n)$, see [4], [6], and [1]. S. Mukai asked to study the geometry of the Grassmannian map associated to any $E \in B(X,n)$ ([4], Problem 4.12). The very first step is to study the geometry of the ruled surface $\mathbf{P}(E)$, which is determined by the following integer $e(E)$.

Let E a stable rank two vector bundle on a smooth curve X of genus g. Take a maximal degree line subbundle L of X and set $e(E) = \deg(E) - 2\cdot\deg(L)$. Hence $e(E) \equiv 0 \pmod 2$. Hence $e(E)$ is even if $\det(E) \cong \omega_X$. A theorem of C. Segre and M. Nagata says that $0 < e(E) \leq g$ ([5]). Set $\rho(g,r,d) := (r+1)d - rg - (r+1)r$. Let τ_g denote the maximal integer r such that $\rho(g,r,g-2) \geq 0$, i.e. the maximal integer such that $\tau_g(\tau_g+1) \leq g$. Hence $\tau_g := \lfloor(-1+\sqrt{4g+1})/2\rfloor$.

Let X be a smooth projective curve of genus $g \geq 3$. We will say that X is a Gieseker-Petri curve if it satisfies Gieseker-Petri's theorem. Such a curve

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is called a Brill-Noether general curve in [1]. A general curve is a Gieseker-Petri curve. R. Lazarfeld proved that many $K3$ surfaces contain Gieseker-Petri curves ([3]).

Here we prove the following result.

Theorem 1. *Let X be a smooth Gieseker-Petri curve of genus $g \geq 5$. Fix an integer r such that $1 \leq r \leq r+1$. There is $E \in B(X, 2r+3)$ such that $e(E) = 2$. For every integer $n \geq 2r+4$ there is no $E \in B(X, 2r+3)$ such that $e(E) = 2$.*

Proof. Fix an integer r such that $1 \leq r \leq \tau_g$. Since X is a Gieseker-Petri curve, $W^r_{g-2}(X)$ is non-empty and irreducible. Fix a general $M \in W^r_{g-2}(X)$. The generality of M and Bril-Noether theory imply $h^0(X, M) = r+1$. However, notice that $h^0(X, R) = \tau_g + 1$ for every $R \in W^{\tau_g}_{g-2}(X)$ by the definition of the integer τ_g. Riemann-Roch and Serre duality give $h^0(X, \omega_X \otimes M^*) = r+2$. The following trick is contained in [2], p. 77. Since $\deg(\omega_X \otimes M^*) = g$, we have $h^0(X, \omega_X^{\otimes 2} \otimes (M^*)^{\otimes 2}) = g+1$. Since $\binom{\tau_g+3}{2} < g+1$, the multiplication map

$$\mu : H^0(X, \omega_X \otimes M^*) \otimes H^0(X, \omega_X \otimes M^*) \to H^0(X, \omega_X^{\otimes 2} \otimes (M^*)^{\otimes 2})$$

is not surjective. Hence its dual

$$\mu^* : H^0(X, \omega_X^{\otimes 2} \otimes (M^*)^{\otimes 2}) \to H^0(X, \omega_X \otimes M^*)^* \otimes H^0(X, \omega_X \otimes M^*)^*$$

has a non-zero kernel. Fix any $\alpha \in \mathrm{Ker}(\mu^*)$. The vector space $H^0(X, \omega_X^{\otimes 2} \otimes (M^*)^{\otimes 2}) = \mathrm{Ext}^1(\omega_X \otimes M^*, M)$ classifies the extensions of $\omega_X \otimes M^*$ by M and μ^* is identified (up to multiplication by a non-zero scalar) with the map

$$\mathrm{Ext}^1(\omega_X \otimes M^*, M) \to \mathrm{Hom}(H^0(X, \omega_X \otimes M^*), H^0(X, M))$$

which send an extension of $\omega_X \otimes M^*$ by M to the connecting homomorphism it determines. Hence α determines a non-trivial extension

$$0 \to M \to E \to \omega_X \otimes M^* \to 0 \tag{1}$$

such that $h^0(X, E) = h^0(X, M) + h^0(X, \omega_X \otimes M^*) = 2r+3$. Since M is general in $W^r_{g-2}(X)$, $\omega_X \otimes M^*$ is general in $W^{r+1}_g(X)$. Hence both M and $\omega_X \otimes M^*$ are spanned. Notice that E is stable if and only if $e(E) = 2$. Assume that E is not stable and take a maximal degree line subbundle L of E. Thus $\deg(L) \geq g-1$ and we have an exact sequence

$$0 \to L \to E \to \omega_X \otimes L^* \to 0 \tag{2}$$

From (1) we get a non-zero map $u : L \to \omega_X \otimes M^*$. Since (1) does not split, u is not an isomorphism. Hence $\deg(L) = g-1$ and there is $P \in X$ such that $L \cong \omega_X \otimes M^*(-P)$. Since $\omega_X \otimes M^*$ is spanned, we get $h^0(X, L) = r+1$. By Rieman-Roch and Serre duality we get $h^0(X, \omega_X \otimes M^*) = r+1$. The exact sequence (2) gives $h^0(X, E) \leq 2r+2$, contradiction. We also checked the last assertion. $\square$

Only the case $r = \tau_g$ of the first part of Theorem 1 is not contained in the range of integers n covered by [6]. However, here we consider arbitrary Gieseker-Petri curves, not just curve with general moduli, and we made the additional information concerning $e(E)$.

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