International Mathematical Forum, 2, 2007, no. 16, 747 - 751

Complemented Subspaces in the Normed Spaces

H. Mazaheri and M. Nasri

Department of Mathematics
Yazd University, Yazd, Iran
hmazaheri@yazduni.ac.ir

Abstract

The purpose of this paper is to introduce and discuss the concept of orthogonality in normed spaces. A concept of orthogonality on normed linear space was introduced. We obtain some subspaces of Banach spaces which are topologically complemented.

Keywords: Banach spaces, Topologically complemented, Chebyshev subspaces, Cochebyshev subspaces

Mathematics Subject Classification: 41A65, 46B50, 46B20, 41A50.

1. Introduction

Suppose X is a normed linear space and $x, y \in X$, x is said to be orthogonal to y and is denoted by $x \perp y$ if and only if $\|x\| \leq \|x + \alpha y\|$ for all scalar α. If M_1 and M_2 are subsets of X, it is defined $M_1 \perp M_2$ if and only if for all $g_1 \in M_1, g_2 \in M_2$, $g_1 \perp g_2$. (see [1]). Let M be a subspace of X, it is defined the set of the metric complemente

$$\hat{M} = \{x \in X : \quad x \perp M\} = \{x \in X : \ \|x\| = \|x + M\|\},$$

and the set of the cometric complemente

$$\breve{M} = \{x \in X : \quad M \perp x\}.$$

We know that a point $g_0 \in M$ is said to be a best approximation (resp. best coapproximation) for $x \in X$ if and only if $\|x - g_0\| = \|x + M\| = dist(x, M)$ (*resp.* $\|g_0 - g\| \leq \|x - g\| \ \forall \ g \in M$). It can be easily proved that g_0 is a best approximation (resp. best coapproximation) for $x \in X$ if and only if $x - g_0 \in \hat{M}$ (*resp.* $x - g_0 \in \breve{M}$). The set of all best approximations (resp.

best coapproximations) of $x \in X$ in M is shown by $P_M(x)$ (*resp.* $R_M(x)$). In other words,

$$P_M(x) = \{g_0 \in M \ : \ x - g_0 \in \hat{M}\}$$

and

$$R_M(x) = \{g_0 \in M \ : \ x - g_0 \in \breve{M}\}.$$

If $P_M(x)$(*resp.* $R_M(x)$) is non-empty for every $x \in X$, then M is called an Proximinal (resp. coproximinal) set. The set M is Chebyshev (resp. cochebyshev) if $P_M(x)$(*resp.* $R_M(x)$) is a singleton set for every $x \in X$.

The problems of coapproximation was initially introduced by Franchetti and Furi ,[1], in order to study some characteristic properties of real Hilbert spaces, and such problems were considered further by Papini and Singer [7], and Rao et. al [8-12]. Also there are some results on coapproximation in [2-5]. In this context, we shall obtain some results on orthogonality and coproximinality subspaces of normed space, and obtain properties on normed spaces that are similar to orthogonality in Hilbert spaces.

Suppose M is a closed subspace of a topological vector space X. If there exists a closed subspace N of X such that

$$X = M + N \ and \ M \bigcap N = \{0\},$$

then M is said to be complemented in X. In this case, X is said to be the direct sum of M and N, and the notation $X = M \oplus N$. Let M be a closed subspace of a topological vector space X. It is well known results that if X is locally convex and $\dim M < \infty$ or $\dim(X/M) < \infty$, then M is complemented in X.

In this section we shall present a list of known lemmas which are needed in the proof of main results.

Lemma 1.1. *Let X be a normed linear space. Let M be a linear subspace of X, and $u \in X \backslash M$. Then the following statements are equivalent:*
1) $u \in \breve{M}$
2) For all $g \in M$, there exists $f_g \in X^*$ such that $\|f_g\| = 1,\ f_g(g) = \|g\|$ and $f_g(u) = 0$.

Lemma 1.2. *Let X be a normed linear space. Let M be a linear subspace of X, and $u \in X \backslash M$. Then the following statements are equivalent:*
1) $u \in \hat{M}$.
2) There exists $f \in X^*$ such that $\|f\| = 1,\ f(u) = \|u\|$ and $f|_M = 0$.

Lemma 1.3 ([4]). *Let M be a closed subspace of a normed linear space X. Then*

i) $M \bigcap \breve{M} = \{0\}$.
ii) $d(g, \breve{M}) = \|g\|$ *for every* $g \in M$.
iii) M *is coproximinal subspace if and only if* $X = M + \breve{M}$.
iv) M *is cochebyshev subspace if and only if* $X = M \oplus \breve{M}$, *where* $\oplus$ *means that the sum decomposition of each element* $x \in X$ *is unique.*

Lemma 1.4 ([13]). *Let* M *be a closed subspace of a normed linear space* X. *Then*
i) $M \bigcap \hat{M} = \{0\}$.
iii) M *is proximinal subspace if and only if* $X = M + \hat{M}$.
iv) M *is Chebyshev subspace if and only if* $X = M \oplus \hat{M}$.

2. Orthogonality complemented

We want characterization the orthogonal complemented subspace in a normed space. In this section suppose X is a normed linear space.

Definition 2.1. *Suppose* M *is a subspace of* X *then* M *is called orthogonal complemented subspace, if either* M *is Chebyshev and* $\hat{M}$ *is a subspace of* X *or* M *is cochebyshev and* $\breve{M}$ *is a subspace of* X.

Lemma 2.2. *Let* X *be a normed space and* M *be a closed subspace of* X. *Then*
a) $\breve{M} \perp M$ *if and only if* $\breve{M} \subseteq \hat{M}$,
b) $M \perp \hat{M}$ *if and only if* $\hat{M} \subseteq \breve{M}$.

Proof. a). For all $x \in \breve{M}$, we have $M \perp x$ if and only if $\breve{M} \subseteq \hat{M}$. Also proof of b) is similar to a). ■

Lemma 2.3. *Let* M *be a closed subspace of an inner product space* X *then* $M^{\perp} = \hat{M} = \breve{M}$. *That is,* M *is orthogonal complemented subspace.*

Theorem 2.4. *Let* X *be a smooth Banach space and* M *be a coproximinal subspace of* X. *Then* M *is an orthogonal complemented subspace of* X.

Proof. Since X is a smooth space, from Lemma 1.1, $\breve{M}$ is a subspace of X. Also $M \cap \breve{M} = \{0\}$, and $\alpha \breve{M} \subseteq \breve{M}$, therefore M is cochebyshev subspace. It follows that M is orthogonal complemented subspace of X. ■

Theorem 2.5. *Let be a normed space and* M *be a proximinal (resp. coproximinal) subspace of* X. *If* $\hat{M}$ *(resp.* $\breve{M}$*) is convex. Then* M *is an orthogonal complemented.*

Proof. We can prove that if M is proximinal (resp. coproximinal) and

$\hat{M}$ (*resp.* $\check{M}$) is convex, then M is a Chebyshev (resp. cochebyshev) subspace of X. Therefore M is orthogonal complemented. ■

There exist a Banach space which is not inner product space and has an orthogonal subspace.

Example 2.6. Let $(M_1, \|.\|_1)$ and $(M_2, \|.\|_2)$ be arbitrary Banach space. Put $X = M_1 \oplus M_2$ with the norm $\|x+y\| = \|x\|_1 + \|y\|_2$. We know that if $M = M_1$ then $\hat{M} = M_2$. Therefore M is Chebyshev and $\hat{M}$ is subspace. It follows that M is an orthogonal complemented subspace.

References

[1] G. Brikhoff, Orthogonality in linear metric space, Duke Math. J. 1(1935) 169-172.

[2] C. Franchetti, M. Furi, Some characteristic properties of real Hilbert spaces, Rwy Romaine Math., 1972.

[3] F. Mazaheri, F. M. Maalek Ghaini, Quasi-orthogonality of the best approximant sets, In press in Nonlinear Analysis, 2006.

[4] H. Mazaheri, S. M. Modarress, Some results concerning proximinality and Coproximinality, Nonlinear Analysis, 62(2005) no. 6, 1123-1126.

[5] H. Mazaheri, S. M. Vaezpour, Orthogonality and ϵ-Orthogonality in Banach spaces, Aust. J. Math. ASnal. Appl., 2 (2005) no. 1, Art. 10, 1-5.

[6] T. D. Narang, Best coapproximation in metric spaces, Publ. Inst. math. (Beograd) (N.S) 51(65)(1992) 71-76.

[7] P. L. Papini, I. Singer, Best coapproximation in normed linear spaces, Mh. Math., 88(1979) 27-44.

[8] G. S. Rao, K. R. Chandrasekaran, Best coapproximation in normed linear spaces with property (Λ), Math. Today, 2(1984) 33-40.

[9] G. S. Rao, K. R. Chandrasekaran, Some properties of the maps R_G and R'_G, Pure Appl. Math. Sci., 23(1986) 21-27.

[10] G. S. Rao, K. R. Chandrasekaran, Characterization of elements of best coapproximation in normed linear spaces, Pure Appl. Math. Sci., 26(1987) 139-147.

[11] G. S. Rao, R. Saravanan, Some results concerning best uniform coapproximation., Journal of Inequal. in pure and Appl. Math., 3(2002) no. 2, 1-13.

[12] G. S. Rao, R. Saravanan, Strongly unique best coapproximation, Kyungpook math. J., 43(2003) no. 4, 519-538.

[13] I. Singer, Best approximation in normal linear spaces by elements of linear subspaces, Springer, New York 1970.

Received: May 6, 2006