International Mathematical Forum, 2, 2007, no. 2, 87 - 96

Adomian decomposition method for nonlinear reaction diffusion system of Lotka-Volterra type

M. Alabdullatif [3] and H. A. Abdusalam [1, 3], E.S. Fahmy [2]

[1]Mathematics Department, Faculty of Science
Cairo University, Giza, Egypt
hosny@operamail.com

[2]King Saud University
Women Students Medical Studies & Sciences Sections
Mathematics Department, P. O. Box 22452, Riyadh 11495
Saudi Arabia
esfahmy@operamail.com

[3]King Saud University
College of Science, Mathematics Department
P.O. Box 2455, Riyadh 11451
Saudi Arabia
mosaadal@ksu.edu.sa

Abstract

In this paper the Adomian decomposition method is used to find an analytic approximate solution for nonlinear reaction diffusion system of Lotka-Volterra type. The results obtained indicate that the method is efficient and accurate.

Mathematics Subject Classification: 35D05, 35J70, 35R35, 49J40
Keywords: Reaction diffusion system, decomposition method, Numerical solution, Lotka-Volterra system

1 Introduction

Recently, some new methods such as extended tanh function method [5,12] extended Jacobi elliptic function expansion method [7], and the simplest equation method [8] have been used in literature to find exact solutions for both partial differential equations and system of partial differential equations. By using the Lie symmetry analysis [4], the exact solution of the nonlinear reaction differential system of the form:

$$u_t = (D(u)u_x)_x + f(u,v), \qquad v_t = (D(v)v_x)_x + g(u,v) \tag{1}$$

is obtained for particular choices of the functions $f(u,v)$ and $g(u,v)$. It is known that this system generalizes a great number of the well-known nonlinear second order evolution systems describing various processes in Physics, Chemistry, and Biology [10,9,11,6]. In this paper we concentrate our work to the following particular case of system (1) which is of Lotka-Volterra type [10] that takes the form:

$$\begin{aligned} u_t &= (uu_x)_x + u(a_1 + b_1 u) + h_1 + c_1 v, \\ v_t &= (vv_x)_x + v(a_2 + b_2 v) + h_2 + c_2 u, \end{aligned} \tag{2}$$

$a_1,\ a_2,\ b_1,\ b_2,\ c_1,\ c_2,\ h_1,\ h_2$ are arbitrary constants such that $b_1 b_2 \neq 0$ and $c_1 c_2 \neq 0$, i.e. system (2) contains quadratic nonlinearity in reaction terms and the two equations are coupled.

A periodic exact solution of system (2) was obtained in [4] and is given by:

$$\begin{aligned} u &= \varphi_0(t) \pm (\varphi_0(t) + \tfrac{2c_1}{b}) \cos\left[\sqrt{\tfrac{b}{2}}x \mp |c_1|t - \beta_0\right], \\ v &= \varphi_0(t) + \tfrac{4c_1}{b} \pm \tfrac{|c_1|}{c_1}(\varphi_0(t) + \tfrac{2c_1}{b}) \sin\left[\sqrt{\tfrac{b}{2}}x \mp |c_1|t - \beta_0\right], \end{aligned} \tag{3}$$

where

$$\varphi_0(t) = \frac{1}{3b}\begin{cases} \frac{2}{t_0 - t} - 6c_1, & a_1 = 3c_1, \\ |3c_1 - a_1| \tanh\left(\frac{|3c_1 - a_1|}{2}(t_0 - t)\right) - a_1 - 3c_1, & a_1 \neq 3c_1, \end{cases} \tag{4}$$

$b_1, b_2, c_1, c_2, h_1, h_2, \beta_0$ and t_0 are arbitrary constants such that, $b = b_1 = b_2 > 0$, $a_2 = a_1 - 6c_1$, $c_2 = -c_1$, $h_1 = (2c_1 a_1 - 6c_1^2)/b$, and $h_2 = h_1 + \frac{4c_1}{b}(3c_1 - a_1)$.

The Adomian decomposition method have been shown to solve easily and more accurately a large class of system of partial differential equations with approximates that converges rapidly to accurate solutions [1,2,3,13]. The implementation of the method has shown reliable results in that few terms are needed to obtain either exact solution or to find an approximate solution of a reasonable degree of accuracy in real physical models. Moreover, no linearization or perturbation is required in the method.

The paper is organized as follows: In Section 2, we present the analysis of the Adomian decomposition method applied to nonlinear coupled system. In Section 3, we apply the Adomian decomposition method to obtain analytic approximate solution for system (2) and the numerical experiment is introduced to obtain the results for comparison purposes. Section 4 is devoted for the conclusions.

2 The analysis of the Adomian decomposition method

In this Section, we introduce the main steps of the Adomian decomposition method [3]. We define the linear operator

$$L_t = \frac{\partial}{\partial t}, \qquad \text{and} \qquad L_t^{-1} = \int_0^t (.)dt, \tag{5}$$

where L_t^{-1} is the inverse operator of L_t.

Using (5), system (2) can be written as:

$$\begin{aligned} L_t u &= (uu_x)_x + u(a_1 + b_1 u) + h_1 + c_1 v, \\ L_t v &= (vv_x)_x + v(a_2 + b_2 v) + h_2 + c_2 u. \end{aligned} \tag{6}$$

Applying the inverse operator to both sides of the above system, we get

$$\begin{aligned} u(x,t) &= f(x) + L_t^{-1}\left[a_1 u + c_1 v + h_1 + F(u)\right], \\ v(x,t) &= g(x) + L_t^{-1}\left[a_2 v + c_2 u + h_2 + G(v)\right], \end{aligned} \tag{7}$$

where

$$F(u) = (uu_x)_x + b_1 u^2, \qquad G(v) = (vv_x)_x + b_2 v^2. \tag{8}$$

are the nonlinear terms in (6), $u(x,0) = f(x)$ and $v(x,0) = g(x)$.

According to the decomposition method [3], we assume that a series solution of the unknown functions $u(x,t)$ and $v(x,t)$ are given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \qquad v(x,t) = \sum_{n=0}^{\infty} v_n(x,t). \tag{9}$$

The nonlinear terms $F(u)$ and $G(v)$ can be decomposed into the infinite series of polynomials given as

$$F(x,t) = \sum_{n=0}^{\infty} A_n, \qquad G(x,t) = \sum_{n=0}^{\infty} B_n. \tag{10}$$

where the components $u_n(x,t)$ and $v_n(x,t)$ will be determined recursively, and the A_n's, B_n's are the so called Adomian Polynomials of u_n's and v_n's respectively.

Specific algorithms were set in [3,13] for calculating Adomian's polynomials for nonlinear terms.

$$A_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} F\left(\sum_{k=0}^{n} \lambda^k u_k\right)\right]_{\lambda=0}, \qquad n \geq 0, \tag{11}$$

$$B_n(v_0, v_1, \cdots, v_n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} G\left(\sum_{k=0}^{n} \lambda^k v\right)\right]_{\lambda=0}, \qquad n \geq 0. \tag{12}$$

Thus,

$$\begin{aligned}
A_0 &= bu_0^2 + u_0 u_{0xx} + (u^2)_x, \\
A_1 &= bu_0 u_1 + u_1 u_{0xx} + u_0 u_{1xx} + 2u_{0x} u_{1x}, \\
A_2 &= \frac{1}{2}[b(2u_1^2 + 4u_0 u_2) + 2u_2 u_{0xx} + u_1 u_{1xx} + \\
&\quad 2u_0 u_{2xx} + 2(u_{1x})^2 + 4u_{0x} u_{2x}], \\
A_3 &= \frac{1}{6}[b(12u_1 u_2 + 12u_0 u_3) + 6u_3 u_{0xx} + \\
&\quad 6u_2 u_{1xx} + 6u_1 u_{2xx} + 6u_0 u_{3xx} + 12u_{1x} u_{2x} + 12u_{0x} u_{3x}], \qquad (13) \\
&\;\;\vdots
\end{aligned}$$

and

$$\begin{aligned}
B_0 &= bv_0^2 + v_0 v_{0xx} + (v^2)_x, \\
B_1 &= bv_0 v_1 + v_1 v_{0xx} + v_0 v_{1xx} + 2v_{0x} v_{1x}, \\
B_2 &= \frac{1}{2}[b(2v_1^2 + 4v_0 v_2) + 2v_2 v_{0xx} + v_1 v_{1xx} + \\
&\quad 2v_0 v_{2xx} + 2(v_{1x})^2 + 4v_{0x} v_{2x}], \\
B_3 &= \frac{1}{6}[b(12v_1 v_2 + 12v_0 v_3) + 6v_3 v_{0xx} + \\
&\quad 6v_2 v_{1xx} + 6v_1 v_{2xx} + 6v_0 v_{3xx} + 12v_{1x} v_{2x} + 12v_{0x} v_{3x}], \qquad (14) \\
&\;\;\vdots
\end{aligned}$$

and so on. The components u_n and v_n for $n \geq 0$ are given by the following recursive relationships:

$$\begin{aligned}
u_0 &= u(x,0) = f(x), \\
v_0 &= v(x,0) = g(x), \\
u_1 &= L_t^{-1}\left[a_1 u_0 + c_1 v_0 + h_1 + A_0\right], \\
v_1 &= L_t^{-1}\left[a_2 v_0 + c_2 u_0 + h_2 + B_0\right], \\
&\;\;\vdots \\
u_{n+1} &= L_t^{-1}\left[a_1 u_n + c_1 v_n + h_1 + A_n\right], \quad n \geq 0, \\
v_{n+1} &= L_t^{-1}\left[a_2 v_n + c_2 u_n + h_2 + B_n\right], \quad n \geq 0. \qquad (15)
\end{aligned}$$

Using the above recursive relationships, we construct the solutions $u(x,t)$ and $v(x,t)$ as

$$u(x,t) = \lim_{n\to\infty} \psi_n(x,t), \qquad v(x,t) = \lim_{n\to\infty} \varphi_n(x,t), \qquad (16)$$

where

$$\psi_n(x,t)=\sum_{i=0}^{n-1}u_i(x,t),\qquad \varphi_n(x,t)=\sum_{i=0}^{n-1}v_i(x,t),\qquad n\geq 1. \tag{17}$$

It is interesting to note that, we obtain the series solution by using the initial conditions only.

3 The test problem for the Adomian decomposition method

The main purpose of the work reported in this section is the testing of the Adomian decomposition based on the method introduced in Section 2. The initial conditions can be obtained from the exact solution (3) and (4). Applying the recurrence relation (15), we get the following:

Case 1: For $a_1=3c_1$, $\varphi_0(t)=\frac{1}{3b}(\frac{2}{t_0-t}-6c_1)$, $a_2=a_1-6c_1$, $h_1=h_2=0$, $c_1=1$, $b_1=b_2=b$, $c_2=-1$,

$$u_0=\frac{1}{3b}(\frac{2}{t_0}-6)+\left(\frac{1}{3b}(\frac{2}{t_0}-6)+\frac{2}{b}\right)\cos\left(\sqrt{\frac{b}{2}}x-\beta_0\right), \tag{18}$$

$$v_0=\frac{1}{3b}(\frac{2}{t_0}-6)+\left(\frac{1}{3b}(\frac{2}{t_0}-6)+\frac{2}{b}\right)\sin\left(\sqrt{\frac{b}{2}}x-\beta_0\right), \tag{19}$$

$$\begin{aligned}
u_1 \;=\;& bt(\frac{2}{t_0}-6c_1+(\frac{2}{t_0}+(\frac{2}{b}-6)c_1)\cos^2\left(\sqrt{\frac{b}{2}}x-\beta_0\right)-\\
&\frac{1}{bt_0^2}(\frac{t}{2}(2t_0c_1+B(2-6t_0c_1))\cos\left(\sqrt{\frac{b}{2}}x-\beta_0\right)(b(2-\\
&6t_0c_1)+(2t_0c_1+b(2-6t_0c_1))\cos\left(\sqrt{\frac{b}{2}}x-\beta_0\right)))+\\
&\frac{bt}{2}(\frac{2}{t_0}+(\frac{2}{b}-6)c_1)^2\sin^2\left(\sqrt{\frac{b}{2}}x-\beta_0\right)+tc_1(\frac{2}{t_0}-\\
&6c_1+\frac{4c_1}{b}+\frac{1}{c_1}((\frac{2}{t_0}+(\frac{2}{b}-6)c_1)|c_1|\sin\left(\sqrt{\frac{b}{2}}x-\beta_0\right)))-\\
&t(\frac{2}{t_0}-6c_1+(\frac{2}{t_0}+(\frac{2}{b}-6)c_1)\cos\left(\sqrt{\frac{b}{2}}x-\beta_0\right))a_1+th_1,
\end{aligned} \tag{20}$$

$$\begin{aligned}
v_1 = {} & \frac{t}{2}(\frac{1}{c_1^2}(b(\frac{2}{t_0}+(\frac{2}{b}-6)c_1)^2|c_1|^2\cos^2\left(\sqrt{\frac{b}{2}}x-\beta_0\right))- \\
& 2c_1(\frac{2}{t_0}-6c_1+(\frac{2}{t_0}+(\frac{2}{b}-6)c_1)\cos\left(\sqrt{\frac{b}{2}}x-\beta_0\right)+ \\
& 2b(\frac{2}{t_0}-6c_1-\frac{4c_1}{b}+\frac{1}{c_1}((\frac{2}{t_0}+(\frac{2}{b}-6)c_1) \\
& |c_1|\sin\left(\sqrt{\frac{b}{2}}x-\beta_0\right)))^2-\frac{1}{bt_0^2c_1^2}((2t_0c_1- \\
& b(2-6t_0c_1))|c_1|\sin\left(\sqrt{\frac{b}{2}}x-\beta_0\right)(c_1(4t_0c_1- \\
& b(2-6t_0c_1))+(2t_0c_1+b(2-6t_0c_1)) \\
& |c_1|\sin\left(\sqrt{\frac{b}{2}}x-\beta_0\right))+2(\frac{2}{t_0}-6c_1-\frac{4c_1}{b}- \\
& \frac{1}{c_1}((\frac{2}{t_0}+(\frac{2}{b}-6)c_1)|c_1|\sin\left(\sqrt{\frac{b}{2}}x-\beta_0\right)))a_2+2h_2, \qquad (21) \\
& \vdots
\end{aligned}$$

In order to prove numerically whether the Adomian decomposition method for system (2) leads to higher accuracy, we evaluate the approximate solution using the 6-terms approximations

$$\psi_6 = u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+u_4(x,t)+u_5(x,t), \qquad (22)$$

$$\varphi_6 = v_0(x,t)+v_1(x,t)+v_2(x,t)+v_3(x,t)+v_4(x,t)+v_5(x,t). \qquad (23)$$

Table 1 and Table 2 shows the exact solution, $u(x,t)$, $v(x,t)$, the approximate solutions, ψ_6, φ_6, the absolute error, $|u-\psi_6|$, $|v-\varphi_6|$ where $t=0.6$, $t_0=4$, $b=2$, $\beta_0=3$, $a_1=3$, $a_2=-3$, $h_1=h_2=0$, $c_1=1$.

Table 1

x_i	u_E	u_A	$\|u_E-u_A\|$
-8	-0.846246	-0.846249	2.64963×10^{-6}
-6	-0.998499	-0.998501	2.50145×10^{-6}
-4	-0.877327	-0.877326	1.56867×10^{-6}
-2	-0.825925	-0.825927	1.96704×10^{-6}
0	-0.989878	-0.989881	3.09441×10^{-6}
2	-0.904823	-0.904822	1.37961×10^{-6}
4	-0.811661	-0.811662	1.21672×10^{-6}
6	-0.974254	-0.974258	3.52984×10^{-6}
8	-0.932091	-0.93209	9.91685×10^{-6}

Table 2

x_i	v_E	v_A	$\lvert v_E - v_A \rvert$
-8	1.17871	1.17871	$1.10767265 \times 10^{-6}$
-6	1.11513	1.11513	$3.43627205 \times 10^{-6}$
-4	1.00315	1.00314	$1.41058665 \times 10^{-6}$
-2	1.15993	1.15993	$1.44739291 \times 10^{-6}$
0	1.14142	1.14142	$2.95697079 \times 10^{-6}$
2	1.00004	1.00004	$2.14922631 \times 10^{-6}$
4	1.13622	1.13622	$1.58285677 \times 10^{-6}$
6	1.16426	1.16426	$2.33107685 \times 10^{-6}$
8	1.00474	1.00474	$2.80561774 \times 10^{-6}$

Case 2: For $a_1 \neq 3c_1$, $\varphi_0(t) = \frac{1}{3b}(|3c_1 - a_1| \tanh\left(\frac{|3c_1-a_1|}{2}(t_0 - t)\right) - a_1 - 3c_1)$. Taking $a_1 = 3$, $h_1 = 1$, $c_1 = 1.2$, $b = 3$, $c_2 = -1.2$ and applying recurrence relation (13) we get

$$\bar{u}_0 = -0.6975 + 0.1024\cos(3 - \sqrt{\frac{3}{2}}x), \tag{24}$$

$$\bar{v}_0 = 0.9024 - 0.1024\sin(3 - \sqrt{\frac{3}{2}}x), \tag{25}$$

$$\begin{aligned}\bar{u}_1 = {} & t(1.4500 - 0.0142)\cos(3 - 1.22x) + \\ & 0.0157\cos^2(3 - 1.22x) - 0.1229\sin(3 - 1.22x) + \\ & 0.0157\sin^2(3 - 1.22x)),\end{aligned} \tag{26}$$

$$\begin{aligned}\bar{v}_1 = {} & t(1.4500 - 0.1229\cos(3 - 1.22x) + \\ & 0.0157\cos^2(3 - 1.22x) + 0.1423\sin(3 - 1.22x) + \\ & 0.0157\sin^2(3 - 1.22x)),\end{aligned} \tag{27}$$

$$\begin{aligned}\bar{u}_2 = {} & t(1 + 0.0107t + t(0.0001 + 0.0036\cos(3 - 1.22x)) \\ & \cos^2(3 - 1.22x) + 0.0107t\sin(3 - 1.22x) - \\ & 0.0020t\sin^2(3 - 1.22x) + t\cos(3 - 1.22x) \\ & (-0.0727 - 0.0021\cos(3 - 1.22x) + (0.0189 + \\ & 2.612 \times 10^{-18}\sin(3 - 1.22x))\sin(3 - 1.22x) + \\ & t\cos(3 - 1.22x)(0.334 - 0.0189\sin(3 - 1.22x) + \\ & 0.0036\sin^2(3 - 1.22x))),\end{aligned} \tag{28}$$

$$\bar{v}_2 = t(1.96 + 0.0107t + t\cos^2(3 - 1.22x)(0.0001 -$$

$$
\begin{aligned}
&0.0036\sin(3-1.22x)+t\cos(3-1.22x)(0.0170+\\
&\cos(3-1.22x)(-0.002-2.612\times10^{-18}\sin(3-1.22x))+\\
&0.0189\sin(3-1.22x)-0.334t\sin(3-1.22x)+\\
&t(0.0001-0.0036\sin(3-1.22x))\sin^2(3-1.22x)+\\
&t\sin(3-1.22x)(0.0727+(0.0184\cos(3-1.22x)-\\
&(0.0048\sin(3-1.22x))),\qquad (29)\\
&\vdots
\end{aligned}
$$

Once again, in order to prove numerically whether the Adomian decomposition method for system (2) leads to higher accuracy, we evaluate the approximate solution using the 6-terms approximations

$$\bar{\psi}_6 = \bar{u}_0(x,t)+\bar{u}_1(x,t)+\bar{u}_2(x,t)+\bar{u}_3(x,t)+\bar{u}_4(x,t)+\bar{u}_5(x,t), \quad (30)$$
$$\bar{\varphi}_6 = \bar{v}_0(x,t)+\bar{v}_1(x,t)+\bar{v}_2(x,t)+\bar{v}_3(x,t)+\bar{v}_4(x,t)+\bar{v}_5(x,t). \quad (31)$$

Table 3 and Table 4 shows the exact solutions u and v. The approximate solutions, $\bar{\psi}_6$, $\bar{\varphi}_6$, the absolute error, $|u-\bar{\psi}_6|$, $|v-\bar{\varphi}_6|$ and the relative error where $t=0.001$, $a_1=3$, $h_1=1$, $c_1=1.2$, $b=3$, $t_0=4$, $\beta_0=3$.

Table 3

x_i	u_E	u_A	$\lvert u_E-u_A\rvert$
-8	-0.59785	-0.59236	5.482×10^{-3}
-6	-0.75921	-0.75373	5.481×10^{-3}
-4	-0.70227	-0.69679	5.481×10^{-3}
-2	-0.62858	-0.62310	5.482×10^{-3}
0	-0.79899	-0.79351	5.480×10^{-3}
2	-0.61029	-0.60480	5.482×10^{-3}
4	-0.73045	-0.72497	5.481×10^{-3}
6	-0.73412	-0.72864	5.481×10^{-3}
8	-0.60830	-0.60282	5.482×10^{-3}

Table 4

x_i	v_E	v_A	$\lvert v_E-v_A\rvert$
-8	0.878821	0.888141	9.321×10^{-3}
-6	0.984272	0.993595	9.323×10^{-3}
-4	0.800109	0.809429	9.320×10^{-3}
-2	0.978233	0.987556	9.323×10^{-3}
0	0.888119	0.897441	9.321×10^{-3}
2	0.848754	0.858075	9.320×10^{-3}
4	0.999483	1.008811	9.323×10^{-3}
6	0.806754	0.816074	9.320×10^{-3}
8	0.952792	0.962114	9.322×10^{-3}

4 Conclusions

We have presented a scheme used to obtain analytic approximate solution of the nonlinear reaction diffusion system of Lotka-Volterra type by using the Adomian decomposition method. The absolute error, exact and numerical results are presented and compared each other in tables for some values of x and fixed t. As expected from the tables the analytic approximate solution clearly indicates that how the Adomian decomposition method obtains efficient results much closer to the accurate solutions.

References

[1] H.A. Abdusalam and M. Alabdulatif, Exact and numerical solutions of the Telegraph model of dispersive variability, Applied Numerical Mathematics, Far East J. Dynamical Systems 8(1). (2006) 65.

[2] H.A. Abdusalam and E.S. Fahmy, Exact solution of the generalized time-delayed Burgers equation through the improved tanh–function method, Mathematical Research Letters, (submitted).

[3] G. Adomian, Solving frontier problems of physics: The decomposition method (Boston, MA: Kluwer Academic (1994).

[4] R. Cherniha, and John R. King, Non-linear reaction-diffusion systems with variable diffusivities: Lie symmetries, ansatz and exact solutions, J. Math. Anal. Appl. 308 (2005) 11.

[5] E.G. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A 277 (2000) 212.

[6] P.C. Fife, Mathematical Aspects of Reacting and diffusing systems, Springer, Berlin, 1979.

[7] Z.T. Fu, S.K. Liu, S.D. Liu and Q. Zhao, New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations, Phys. Lett. A 290 (2001) 72.

[8] N.A. Kudryashov, Simplest equation method to look for exact solutions of nonlinear differential equations, Chaos, Solitons and Fractals, 24 (2005) 1217.

[9] J.D. Murray, Nonlinear differential equation models in Biology, Clarendon Press, Oxford, 1977.

[10] J.D. Murray, Mathematical Biology I. An introduction. Third Edition, Springer-Verlag Berlin Heidelberg (2003)

[11] J.D. Murray, Mathematical Biology, Springer, Berlin, 1989..

[12] A.M. Wazwaz, The tanh-method: solitons and periodic solutions for the Dodd-Bullough-Mikhailov and the Tzitzeica-Dodd-Bullough equations, Chaos, Solitons and Fractals 25 (2005) 55.

[13] A.M. Wazwaz, Partial differential equations: Methods and applications, Balkema Publishers, The Netherlands, 2002.

Received: April 12, 2006