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INTUITIONISTIC FUZZY GENERALIZED BI-IDEALS OF A SEMIGROUP

Kul Hur

Division of Mathematics and Informational Statistics
Wonkwang University, Iksan, Chonbuk, Korea 579-792
kulhur@wonkwang.ac.kr

Hee Won Kang

Dept. of Mathematics Education
Woosuk University
Hujong-Ri Samrae-Eup, Wanju-kun Chonbuk, Korea 565-701
khwon@woosuk.ac.kr

Jang Hyun Ryou

Bong Young Girl′s Middle School 325-8
Shinjung, 7 dong, Yangchongu, Seoul, Korea
donggni@hanmail.net

Abstract. We introduce the concept of an intuitionistic fuzzy generalized bi-ideal of a semigroup, which is an extension of the concept of an intuitionitic fuzzy bi-ideal(and of a nonintuitionistic fuzzy bi-ideal and a nonintuitionistic fuzzy ideal of a semigroup), and characterize regular semigroups, and both intraregularand left quasiregular semigroup in terms of intuitinistic fuzzy generalized bi-ideals.

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1. Introduction

In his pioneering paper[23], Zadeh introduced the notion of a fuzzy set in a set X as a mapping from X into the closed unit interval $[0, 1]$. Since then, Rosenfeld[22] and Liu[21] applied this concept to group theory.

In 1986, Atanassov[1] introduced the concept of intuitionistic fuzzy sets as the generalization of fuzzy sets. After that time, Çoker and his colleagues[6,7,8], Lee and Lee[20], and Hur and his colleagues[13] introduced the concept of intuitionistic fuzzy topological spaces using using intuitionistic fuzzy sets and investigated some of its properties. In particular, Hur and his colleagues [12] applied the notion of intuitionistic fuzzy set to topological group. In 1989, Biswas[3] introduced the concept of intuitionistic fuzzy subgroups and studied some of it's properties. In 2003, Banerjee and Basnet[2] investigated intuitionistic fuzzy subrings and intuitionistic fuzzy ideals using intuitionistic fuzzy sets. Also, Hur and his colleagues[9,11,14-16] studied various properties of intuitionistic fuzzy subgroupoids, intuitionistic fuzzy subgroups, intuitionistic fuzzy subrings, intuitionistic fuzzy ideals(filters) and intuitionistic fuzzy congruences.

In [19], Lajos charicterized semigroups, which are regular, and both intraregular and left quasiregular, in terms of generalized bi-ideals.

In this paper, we will introduce the concept of an intuitionistic fuzzy generalized bi-ideal of a semigroup, which is an extension of the notion of a nonintuitionistic fuzzy generalized bi-ideal (and of a nonintuitionistic fuzzy bi-ideal and a nonintuitionistic fuzzy ideal), and characterize such semigroups by intuitionistic fuzzy generalized bi-ideals.

For other charactarizations of semigroups by intuitionistic fuzzy bi-ideals and intuitionistic fuzzy ideals, see [14].

2. Preliminaries

We will list some concept and two results needed in the later sections and we obtain some results.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I and for an ordinary subset of a set X, we will denote the characteristic function of A as χ_A.

Definition 2.1[1,6]. *Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an intuitionistic fuzzy set(in short, IFS) in X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership(namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the fuzzy empty set and the intuitionistic fuzzy whole set in a set X defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$ for each $x \in X$, respectively.*

We will denote the set of all IFSs in X as IFS(X).

Definition 2.2[1]. *Let X be a nonempty sets and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be an IFSs in X. Then*

(1) $A \subset B$ if and only if $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ if and only if $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $[\ ]A = (\mu_A, 1 - \mu_A)$, $<\ \ > A = (1 - \nu_A, \nu_A)$.

Definition 2.3[6]. *Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then*

(1) $\bigcap A_i = (\wedge \mu_{A_i}, \vee \nu_{A_i})$.

(2) $\bigcup A_i = (\vee\mu_{A_i}, \wedge\nu_{A_i})$.

Definition 2.4[9]. *Let* $(X, \cdot)$ *be a groupoid and let* $A, B \in IFS(X)$. *Then the intuitionistic fuzzy product of* A *and* B, $A \circ B$ *is defined as follows : for each* $x \in X$,

$$A \circ B(x) = \begin{cases} (\bigvee_{x=yz}[\mu_A(y) \wedge \mu_B(z)], \bigwedge_{x=yz}[\nu_A(y) \vee \nu_B(z)]) & \text{if } x = yz, \\ (0, 1) & \text{if otherwise.} \end{cases}$$

It is clear that for any $A, B, C \in \mathrm{IFS}(X)$, if $B \subset C$, then $A \circ B \subset A \circ C$ and $B \circ A \subset C \circ A$.

Let S be a semigroup. By a *subsemigroup* of S we mean a non-empty subset of A of such that

$$A^2 \subset A$$

and by a *left* [resp. *right*] *ideal* of S we mean a non-empty subset A of S such that

$$SA \subset A \quad [\text{resp. } AS \subset A].$$

By *tow-sided ideal* or simply *ideal* we mean a subset A of S which is both a left and a right ideal of S. We well denote the set of all left ideals [resp. right ideals and ideals] of S as $\mathrm{LI}(S)$ [resp. $\mathrm{RI}(S)$ and $\mathrm{I}(S)$].

Definition 2.5[9]. *Let* S *be a semigroup and let* $A \in IFS(S)$. *Then* A *is called an :*

(1) intuitionistic fuzzy subsemigroup (in short, IFSG) of S *if*

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \quad \text{and} \quad \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$$

for any $x, y \in S$,

(2) intuitionistic fuzzy left ideal (in short, IFLI) of S *if*

$$\mu_A(xy) \geq \mu_A(y) \quad \text{and} \quad \nu_A(xy) \leq \nu_A(y)$$

for any $x, y \in S$,

(3) intuitionistic fuzzy right ideal (in short, IFRI) of S *if*

$$\mu_A(xy) \geq \mu_A(x) \quad \text{and} \quad \nu_A(xy) \leq \nu_A(x)$$

for any $x, y \in S$,

(4) intuitionistic fuzzy (two-sided) ideal (in short, IFI) of S *if it is both an intuitionistic fuzzy left and an intuitionistic fuzzy right ideal of* S.

We will denote the set of all IFSGs [resp. IFLIs, IFRIs and IFIs] of S as IFSG(S) [resp. IFLI(S), IFRI(S) and IFI(S)]. It is clear that $A \in$ IFI(S) if and only if $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$ for any $x, y \in S$, and if $A \in$ IFLI(S) [resp. IFRI(S) and IFI(S)], then $A \in$ IFSG(S).

Result 2.A[9, Definition 3.1 and Proposition 3.2]. *Let* S *be a semigroup and let* $0_\sim \neq A \in$ *IFS*(S). *Then* $A \in$ *IFSG*(S) *if and only if* $A \circ A \subset A$.

Result 2.B[9, Proposition 3.8]. *Let* A *be a non-empty subset of a semigroup* S.

(1) A *is a subsemigroup of* S *if and only if* $(\chi_A, \chi_{A^c}) \in$ *IFSG*(S).

(2) $A \in$ *LI*(S) *[resp. RI*(S) *and I*(S)*] if and only if* $(\chi_A, \chi_{A^c}) \in$ *IFLI*(S) *[resp. IFRI*(S) *and IFI*(S)*].*

Lemma 2.6. *Let* S *be a semigroupiod let* $A \in$ *IFS*(S). *Then* $A \in$ *IFLI*(S) *if and only if* $1_\sim \circ A \subset A$.

Proof. $(\Rightarrow)$: Suppose $A \in$ IFLI(S) and let $a \in S$.

Case (i) : Suppose $(1_\sim \circ A)(a) = (0, 1)$. Then clearly $1_\sim \circ A \subset A$.

Case (ii) : Suppose $(1_\sim \circ A)(a) \neq (0, 1)$. Then there exist $x, y \in S$ with $a = xy$. Thus

$$\begin{aligned}
\mu_{1_\sim \circ A}(a) &= \bigvee_{a=xy} [\mu_{1_\sim}(x) \wedge \mu_A(y)] \\
&\leq \bigvee_{a=xy} [1 \wedge \mu_A(xy)] \qquad (\text{Since } A \in \text{ IFLI}(S)) \\
&= \bigvee_{a=xy} [1 \wedge \mu_A(a)] = \mu_A(a)
\end{aligned}$$

and

$$\begin{aligned}\nu_{1_\sim \circ A}(a) &= \bigwedge_{a=xy}[\nu_{1_\sim}(x) \vee \nu_A(y)] \geq \bigwedge_{a=xy}[0 \vee \nu_A(xy)] \\ &= \bigwedge_{a=xy}[0 \vee \nu_A(a)] = \nu_A(a).\end{aligned}$$

Hence, in all, $1_\sim \circ A \subset A$.

($\Leftarrow$) : Suppose the necessary condition holds. Let $A \in \mathrm{IFS}(S)$ and let $a = xy$ for any $x, y \in S$. Then, by the hypothesis, $1_\sim \circ A \subset A$. Thus

$$\begin{aligned}\mu_A(xy) &= \mu_A(a) \geq \mu_{(1_\sim \circ A)}(a) = \bigvee_{a=bc}[\mu_{1_\sim}(b) \wedge \mu_A(c)] \\ &\geq \mu_{1_\sim}(x) \wedge \mu_A(y) \qquad (\text{Since } a = xy) \\ &= 1 \wedge \mu_A(y) = \mu_A(y)\end{aligned}$$

and

$$\begin{aligned}\nu_A(xy) &= \nu_A(a) \leq \nu_{(1_\sim \circ A)}(a) = \bigwedge_{a=bc}[\nu_{1_\sim}(b) \vee \nu_A(c)] \\ &\leq \nu_{1_\sim}(x) \vee \nu_A(y) = 0 \vee \nu_A(y) = \nu_A(y).\end{aligned}$$

Hence $A \in \mathrm{IFLI}(S)$. This completes the proof. ■

Lemma 2.6$'$[The dual of Lemma 1.6]. *Let S be a semigroup and let $A \in$ IFS(S). Then $A \in$ IFRI(S) if and only if $A \circ 1_\sim \subset A$.*

The combined effect of these two lemmas is as follows :

Theorem 2.7. *Let S be a semigroup and let $A \in$ IFS(S). Then $A \in$ IFI(S) if and only if $1_\sim \circ A \subset A$ and $A \circ 1_\sim \subset A$.*

3. Intuitionistic fuzzy generalized bi-ideals.

A subsemigroup A of a semigroup S is called a *bi-ideal* of S if $ASA \subset A$. We will denote the set of all bi-ideals of S as $\mathrm{BI}(S)$.

Definition 3.1[14]. *Let S be a semigroup and let $A \in IFSG(S)$. Then A is called an intuitionistic fuzzy bi-ideal (in short, IFBI) of S if*

$$\mu_A(xyz) \geq \mu_A(x) \wedge \mu_A(z) \quad and \quad \nu_A(xyz) \leq \nu_A(x) \vee \nu_A(z)$$

for any $x, yz \in S$.

We will denote the set of all IFBIs of S as IFBI(S).

Result 3.A[14, Proposition 2.5]. *Let A be a non-empty subset of a semigroup S. Then $A \in BI(S)$ if and only if $(\chi_A, \chi_{A^c}) \in IFBI(S)$.*

Remark 3.2. *Let S be a semigroup.*

(1) If μ_A is a fuzzy left ideal [resp. right ideal ideal and bi-ideal] of S, then $A = (\mu_A, \mu_A^c) \in IFLI(S)$ [resp. $IFRI(S)$, $IFI(S)$ and $IFBI(S)$].

(2) If $A \in IFBI(S)$, then μ_A and ν_A^c are fuzzy bi-ideals of S.

(3) If $A \in IFBI(S)$, then $[\ ]A, \langle\ \rangle A \in IFBI(S)$.

A nonempty subset A of a semigroup S is called a *generalized bi-ideal* [19] if $ASA \subset A$. We will denote the set of all generalized bi-ideals of S as GBI(S).

Definition 3.3. *Let S be a semigroup and let $A \in IFS(S)$. Then A is called an intuitionistic fuzzy generalized bi-ideal(in short, IFGBI) of S if for any $x, y, z \in S$,*

$$\mu_A(xyz) \geq \mu_A(x) \wedge \mu_A(z) \text{ and } \nu_A(xyz) \leq \nu_A(x) \vee \nu_A(z).$$

We will denote the set of all IFGBIs of S as IFGBI(S). It is clear that IFBI(S) $\subset$ IFGBI(S). But the converse inclusion does not hold in general.

Example 3.4. *Let $S = \{a, b, c, d\}$ be the semigroup with the following multiplication table:*

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

We define a complex mapping $A : S \to I \times I$ as follows :

$$A(a) = (0.5, 0.4),\ A(b) = (0, 1),\ A(c) = (0.2, 0.8),\ A(d) = (0, 1).$$

Then we can easily show that $A \in$ IFGBI(S) but $A \notin$ IFBI(S). ■

Remark 3.5. *Let S be a semigroup.*

(1) If μ_A is a fuzzy generalized bi-ideal of S, then $A = (\mu_A, \mu_{A^c}) \in$ IFGBI(S).

(2) If $A \in$ IFGBI(S), then μ_A and ν_{A^c} are fuzzy generalized bi-ideals of S.

(3) If $A \in$ IFGBI(S), then $[\ ]A$, $\langle\ \rangle A \in$ IFGBI(S).

The following two lemmas are easily seen.

Lemma 3.6. *Let A be a nonempty subset of a semigroup S. Then $A \in$ GBI(S) if and only if $(\chi_A, \chi_{A^c}) \in$ IFGBI(S).*

Lemma 3.7. *Let S be a semigroup and let $A \in$ IFS(S). Then $A \in$ IFGBI(S) if and only if $A \circ 1_\sim \circ A \subset A$.*

4. Regular semigroups.

A semigroup S is said to be *regular* if for each $a \in S$, there exists an $x \in S$ such that $a = axa$.

Proposition 4.1. *Let S be a regular semigroup. Then IFGBI(S) ⊂IFBI(S).*

Proof. Let $A \in$ IFGBI(S) and let $a, b \in S$. Since S is regular, there exists an $x \in S$ such that $b = bxb$. Then

$$\mu_A(ab) = \mu_A(a(bxb)) = \mu_A(a(bx)b) \geq \mu_A(a) \wedge \mu_A(b)$$

and

$$\nu_A(ab) = \nu_A(a(bxb)) = \nu_A(a(bx)b) \leq \nu_A(a) \vee \nu_A(b).$$

Thus $A \in \mathrm{IFSG}(S)$. So $A \in \mathrm{IFBI}(S)$. Hence $\mathrm{IFGBI}(S) \subset \mathrm{IFBI}(S)$. ■

Theorem 4.2. *Let S be a semigroup. Then S is regular if and only if $A = A \circ 1_\sim \circ A$ for each $A \in$ IFGBI(S).*

Proof. $(\Rightarrow)$: Suppose S is regular. Let $A \in \mathrm{IFGBI}(S)$ and let $a \in S$. Since S is regular, there exists an $x \in S$ such that $a = axa$. Then

$$\begin{aligned}
\mu_{A\circ 1_\sim \circ A}(a) &= \bigvee_{a=yz} [\mu_{A\circ 1_\sim}(y) \wedge \mu_A(z)] \\
&\geq \mu_{A\circ 1_\sim}(ax) \wedge \mu_A(a) \qquad \text{(Since } a = axa) \\
&= (\bigvee_{ax=pq} \mu_A(p) \wedge \mu_{1_\sim}(q)) \wedge \mu_A(a) \\
&\geq \mu_A(a) \wedge \mu_{1_\sim}(x) \wedge \mu_A(a) \\
&= \mu_A(a) \wedge 1 \wedge \mu_A(a) = \mu_A(a)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{A\circ 1_\sim \circ A}(a) &= \bigwedge_{a=yz} [\nu_{A\circ 1_\sim}(y) \vee \nu_A(z)] \leq \nu_{A\circ 1_\sim}(ax) \vee \nu_A(a) \\
&= (\bigwedge_{ax=pq} \nu_A(p) \vee \nu_{1_\sim}(q)) \vee \nu_A(a) \leq \nu_A(a) \vee \nu_{1_\sim}(x) \vee \nu_A(a) \\
&= \nu_A(a) \vee 1 \vee \nu_A(a) = \nu_A(a).
\end{aligned}$$

Thus $A \subset A \circ 1_\sim \circ A$. Since $A \in \mathrm{IFGBI}(S)$, by Lemma 3.7, $A \circ 1_\sim \circ A \subset A$. Hence $A = A \circ 1_\sim \circ A$.

$(\Leftarrow)$: Suppose the necessary condition holds. Let $A \in \mathrm{GBI}(S)$. There, by Lemma 3.6, $(\chi_A, \chi_{A^c}) \in \mathrm{IFGBI}(S)$. Thus, by the hypothesis,

$$(\chi_A, \chi_{A^c}) \circ 1_\sim \circ (\chi_A, \chi_{A^c}) = (\chi_A, \chi_{A^c}).$$

Let $a \in S$. Then

$$\begin{aligned}
\mu_{(\chi_A,\chi_{A^c})\circ 1_\sim \circ (\chi_A,\chi_{A^c})}(a) &= \bigvee_{a=yz} [\mu_{(\chi_A,\chi_{A^c})\circ 1_\sim}(y) \wedge \chi_A(z)] \\
&= \chi_A(a) = 1
\end{aligned}$$

and

$$\begin{aligned}\nu_{(\chi_A,\chi_{A^c})\circ 1_\sim\circ(\chi_A,\chi_{A^c})}(a) &= \bigwedge_{a=yz}[\nu_{(\chi_A,\chi_{A^c})\circ 1_\sim}(y)\vee\chi_{A^c}(z)]\\ &= \chi_{A^c}(a)=0.\end{aligned}$$

Thus there exist $b, c \in S$ with $a = bc$ such that

$$\mu_{(\chi_A,\chi_{A^c})\circ 1_\sim}(b)=\chi_A(c)=1 \text{ and } \nu_{(\chi_A,\chi_{A^c})\circ 1_\sim}(b)=\chi_{A^c}(c)=0.$$

So $\bigvee_{b=pq}[\chi_A(p)\wedge\mu_{1_\sim}(q)]=1$ and $\bigwedge_{b=pq}[\chi_{A^c}(p)\vee\nu_{1_\sim}(q)]=0$. Then there exist $d, e \in S$ with $b = de$ such that

$$\chi_A(d)=\mu_{1_\sim}(e)=1 \text{ and } \chi_{A^c}(d)=\nu_{1_\sim}(e)=0.$$

Thus $d \in A, e \in S, c \in S$ and $a = bc = (de)c \in ASA$. So $A \subset ASA$. Since $A \in$ GBI(S), it is clear that $ASA \subset A$. Hence $A = ASA$. Therefore A is regular. This completes the proof. ■

The following result is due to Lemma 3.7 and Theorem 4.2.

Theorem 4.3. *A semigroup S is regular if and only if (IFGBI(S), 0) is a regular semigroup.*

Theorem 4.4. *A semigroup S is regular if and only if for each $A \in$ IFGBI(S) and each $B \in$ IFI(S), $A \cap B = A \circ B \circ A$.*

Proof. ($\Rightarrow$) : Suppose S is regular. Let $A \in$ IFGBI(S) and let $B \in$ IFI(S). Then, by Lemma 3.7, $A \circ B \circ A \subset A \circ 1_\sim \circ A \subset A$. Also, by Theorem 2.7, $A \circ B \circ A \subset 1_\sim \circ B \circ 1_\sim \subset 1_\sim \circ B \subset B$. So $A \circ B \circ A \subset A \cap B$. Now let $a \in S$. Since S is regular, there exists an $x \in S$ such that $a = axa (= axaxa)$. Since $B \in$ IFI(S),

$$\mu_B(xax)\geq\mu_B(ax)\geq\mu_B(a) \text{ and } \nu_B(xax)\leq\nu_B(ax)\leq\nu_B(a).$$

Then

$$\mu_{A\circ B\circ A}(a) = \bigvee_{a=yz}[\mu_A(y)\wedge\mu_{B\circ A}(z)]$$

$$
\begin{aligned}
&\geq \mu_A(a) \wedge \mu_{B\circ A}(xaxa) \qquad \text{(Since } a = axaxa\text{)} \\
&= \mu_A(a) \wedge \bigvee_{xaxa=pq} [\mu_B(p) \wedge \mu_A(p)] \\
&\geq \mu_A(a) \wedge \mu_B(xax) \wedge \mu_A(a) \\
&\geq \mu_A(a) \wedge \mu_B(a) \wedge \mu_A(a) \\
&= \mu_A(a) \wedge \mu_B(a) = \mu_{A\cap B}(a)
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_{A\circ B\circ A}(a) &= \bigwedge_{a=yz} [\nu_A(y) \vee \nu_{B\circ A}(z)] \leq \nu_A(a) \vee \nu_{B\circ A}(xaxa) \\
&= \nu_A(a) \vee \bigwedge_{xaxa=pq} [\nu_B(p) \vee \nu_A(p)] \leq \nu_A(a) \vee \nu_B(xax) \vee \nu_A(a) \\
&\leq \nu_A(a) \vee \nu_B(a) \vee \nu_A(a) = \nu_A(a) \vee \nu_B(a) = \nu_{A\cap B}(a).
\end{aligned}
$$

So $A \cap B \subset A \circ B \circ A$. Hence $A \circ B \circ A = A \cap B$.

($\Leftarrow$) : Suppose the necessary condition holds. It is clear that $1_\sim \in \text{IFI}(S)$. Let $A \in \text{IFGBI}(S)$. Then, by the hypothesis, $A = A \cap 1_\sim = A \circ 1_\sim \circ A$. Hence, by Theorem 4.2, S is regular. This completes the proof. ∎

Result 4.A[19, Theorems 1 and 4]. *Let S be a semigroup. Then the following are equivalent:*

(1) S is regular.

(2) $A \cap L \subset AL$ for each $A \in$ GBI(S) and each $L \in$ LI(S).

(3) $R \cap A \cap L \subset RAL$ for each $A \in$GBI(S), each $L \in$ LI(S) and each $R \in$ RI(S).

Now we give a characterization of a regular semigroup in terms of intuitionistic fuzzy generalized bi-ideals and intuitionistic fuzzy bi-ideals.

Theorem 4.5. *Let S be a semigroup. Then the following are equivalent:*

(1) S is regular.

(2) $A \cap B \subset A \circ B$ for each $A \in$ IFBI(S) and each $B \in$ IFLI(S).

(3) $A \cap B \subset A \circ B$ for each $A \in$ IFGBI(S) and each $B \in$ IFLI(S).

(4) $C \cap A \cap B \subset C \circ A \circ B$ for each $A \in$ IFBI(S), each $B \in$ IFLI(S) and each $C \in$ IFRI(S).

(5) $C \cap A \cap B \subset C \circ A \circ B$ for each $A \in$ IFGBI(S) and each $B \in$ IFRI(S).

Proof. (1) $\Rightarrow$ (2): Suppose S is regular. Let $A \in$ IFBI(S), let $B \in$ IFLI(S) and let $a \in S$. Since S is regular, there exists an $x \in S$ such that $a = axa$. Then $(A \circ B)(a) \neq (0, 1)$. Thus

$$\begin{aligned}\mu_{A\circ B}(a) &= \bigvee\nolimits_{a=yz}[\mu_A(y) \wedge \mu_B(z)] \\ &\geq \mu_A(a) \wedge \mu_B(xa) && \text{(Since } a = axa) \\ &\geq \mu_A(a) \wedge \mu_B(a) && \text{(Since } B \in \text{IFLI}(S)) \\ &= \mu_{A\cap B}(a)\end{aligned}$$

and

$$\begin{aligned}\nu_{A\circ B}(a) &= \bigwedge\nolimits_{a=yz}[\nu_A(y) \vee \nu_B(z)] \leq \nu_A(a) \vee \nu_B(xa) \\ &\leq \nu_A(a) \vee \nu_B(a) = \nu_{A\cap B}(a).\end{aligned}$$

Hence $A \circ B \subset A \cap B$.

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (1): Suppose the condition (3) holds. Let $A \in$GBI(S), let $L \in$LI(S) and let $a \in A \cap L$. Then $a \in A$ and $a \in L$. Since $A \in$ GBI(S), by Lemma 3.6, $(\chi_A, \chi_{A^c}) \in$ IFGBI(S). By Result 2.B(2), $(\chi_L, \chi_{L^c}) \in$ IFLI(S). Thus, by the hypothesis,

$$(\chi_A, \chi_{A^c}) \cap (\chi_L, \chi_{L^c}) \subset (\chi_A, \chi_{A^c}) \circ (\chi_L, \chi_{L^c}).$$

So

$$\mu_{(\chi_A,\chi_{A^c})\circ(\chi_L,\chi_{L^c})}(a) \geq \mu_{(\chi_A,\chi_{A^c})\cap(\chi_L,\chi_{L^c})}(a) = \chi_A(a) \wedge \chi_L(a) = 1$$

and

$$\nu_{(\chi_A,\chi_{A^c})\circ(\chi_L,\chi_{L^c})}(a) \leq \nu_{(\chi_A,\chi_{A^c})\cap(\chi_L,\chi_{L^c})}(a) = \chi_{A^c}(a) \vee \chi_{L^c}(a) = 0.$$

Then $[(\chi_A, \chi_{A^c}) \circ (\chi_L, \chi_{L^c})](a) \neq (0, 1)$. Thus

$$\bigvee\nolimits_{a=yz}[\chi_A(y) \wedge \chi_L(z)] = 1 \text{ and } \bigwedge\nolimits_{a=yz}[\chi_{A^c}(y) \vee \chi_{L^c}(z)] = 0.$$

So there exist $b, c \in S$ with $a = bc$ such that

$$\chi_A(b) = 1,\ \chi_{A^c}(b) = 0 \text{ and } \chi_L(c) = 1,\ \chi_{L^c}(c) = 0.$$

Thus $b \in A$ and $c \in L$, i.e., $a = bc \in AL$. So $A \cap L \subset AL$. Hence, by Result 4.A, S is regular.

(1) $\Rightarrow$ (4): Suppose S is regular. Let $A \in \mathrm{IFBI}(S)$, let $B \in \mathrm{IFLI}(S)$ and let $C \in \mathrm{IFRI}(S)$. Since S is regular, there exists an $x \in S$ such that $a = axa$. Then

$$\begin{aligned}
\mu_{C\circ A\circ B}(a) &= \textstyle\bigvee_{a=yz}[\mu_C(y) \wedge \mu_{A\circ B}(z)] \\
&\geq \mu_C(ax) \wedge \mu_{A\circ B}(a) && \text{(Since } a = axa\text{)} \\
&\geq \mu_C(a) \wedge (\textstyle\bigvee_{a=pq}[\mu_A(p) \wedge \mu_B(q)]) && \text{(Since } C \in \mathrm{IFRI}(S)\text{)} \\
&\geq \mu_C(a) \wedge \mu_A(a) \wedge \mu_B(xa) && \text{(Since } a = axa\text{)} \\
&\geq \mu_C(a) \wedge \mu_A(a) \wedge \mu_B(a) && \text{(Since } C \in \mathrm{IFRI}(S)\text{)} \\
&= \mu_{C\cap A\cap B}(a)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{C\circ A\circ B}(a) &= \textstyle\bigwedge_{a=yz}[\nu_C(y) \vee \nu_{A\circ B}(z)] \leq \nu_C(ax) \vee \nu_{A\circ B}(a) \\
&\leq \nu_C(a) \vee (\textstyle\bigwedge_{a=pq}[\nu_A(p) \vee \nu_B(q)]) \leq \nu_C(a) \vee \nu_A(a) \vee \nu_B(xa) \\
&\leq \nu_C(a) \vee \nu_A(a) \vee \nu_B(a) = \nu_{C\cap A\cap B}(a).
\end{aligned}$$

Hence $C \cap A \cap B \subset C \circ A \circ B$.

(4) $\Rightarrow$ (5): It is clear.

(5) $\Rightarrow$ (1): Suppose the condition (5) holds. Let $A \in \mathrm{GBI}(S)$, let $B \in \mathrm{LI}(S)$ and let $R \in \mathrm{RI}(S)$. Let $a \in R \cap A \cap L$. Then $a \in R$, $a \in A$ and $a \in L$. Since $A \in \mathrm{GBI}(S)$, by Lemma 3.6, $(\chi_A, \chi_{A^c}) \in \mathrm{IFGBI}(S)$. By Result 2.B(2), $(\chi_R, \chi_{R^c}) \in \mathrm{IFRI}(S)$ and $(\chi_L, \chi_{L^c}) \in \mathrm{IFLI}(S)$. By the hypothesis,

$$(\chi_R, \chi_{R^c}) \cap (\chi_A, \chi_{A^c}) \cap (\chi_L, \chi_{L^c}) \subset (\chi_R, \chi_{R^c}) \circ (\chi_A, \chi_{A^c}) \circ (\chi_L, \chi_{L^c}).$$

Then

$$\begin{aligned}
\mu_{(\chi_R,\chi_{R^c})\circ(\chi_A,\chi_{A^c})\circ(\chi_L,\chi_{L^c})}(a) &\geq \mu_{(\chi_R,\chi_{R^c})\cap(\chi_A,\chi_{A^c})\cap(\chi_L,\chi_{L^c})}(a) \\
&= \chi_R(a) \wedge \chi_A(a) \wedge \chi_L(a) = 1
\end{aligned}$$

and

$$\begin{aligned}
\nu_{(\chi_R,\chi_{R^c})\circ(\chi_A,\chi_{A^c})\circ(\chi_L,\chi_{L^c})}(a) &\leq \nu_{(\chi_R,\chi_{R^c})\cap(\chi_A,\chi_{A^c})\cap(\chi_L,\chi_{L^c})}(a) \\
&= \chi_R(a) \vee \chi_A(a) \vee \chi_L(a) = 0.
\end{aligned}$$

Thus $(\chi_R, \chi_{R^c}) \circ (\chi_A, \chi_{A^c}) \circ (\chi_L, \chi_{L^c}) \neq (0, 1)$. So

$$\bigvee_{a=yz}[\mu_{(\chi_R,\chi_{R^c})\circ(\chi_A,\chi_{A^c})}(y) \wedge \chi_L(z)] = 1$$

and

$$\bigwedge_{a=yz}[\nu_{(\chi_R,\chi_{R^c})\circ(\chi_A,\chi_{A^c})}(y)\vee\chi_L(z)]=0.$$

Then there exist $b, c \in S$ with $a = bc$ such that

$$\mu_{(\chi_R,\chi_{R^c})\circ(\chi_A,\chi_{A^c})}(b)=1,\ \nu_{(\chi_R,\chi_{R^c})\circ(\chi_A,\chi_{A^c})}(b)=0$$

and

$$\chi_L(c)=1 \text{ and } \chi_{L^c}=0. \qquad (*)$$

Thus $[(\chi_R,\chi_{R^c})\circ(\chi_A,\chi_{A^c})](b)\neq(0,1)$. So

$$\bigvee_{b=pq}[\chi_R(p)\wedge\chi_A(q)]=1 \text{ and } \bigwedge_{b=pq}[\chi_{R^c}(p)\vee\chi_{A^c}(q)]=0.$$

Then there exist $d, e \in S$ with $b = de$ such that

$$\chi_R(d)=1,\ \chi_{R^c}(d)=0 \text{ and } \chi_A(e)=1,\ \chi_{A^c}(e)=0. \qquad (**)$$

By $(*)$ and $(**)$, $d \in R$, $e \in A$ and $c \in L$. Thus $a = bc = dec \in RAL$. So $R \cap A \cap L\cap \subset RAL$. Hence, by Result 4.A, S is regular. This complete the proof. ■

5. Left quasiregular semigroups.

A semigroup S is said to be *left quasiregular* if every left ideal of S is globally idempotent.

Result 5.A[4, Proposition 1.1]. *A semigroup S is left quasiregular if and only if for each $a \in S$, there exist $x, y \in S$ such that $a = xaya$.*

The following result can be easily proved.

Lemma 5.1. *Let S be a semigroup. If S is left quasiregular, then IFGBI(S) = IFBI(S), i.e., IFGBI$(S) \subset$ IFBI(S).*

Theorem 5.2. *Let S be a semigroup. Then S is left quasiregular if and only if $A \circ A = A$ for each $A \in$ IFLI(S).*

Proof. ($\Rightarrow$): Suppose S is left quasiregular and let $A \in \mathrm{IFLI}(S)$. Then , by Definition 2.5 and Result 2.A, $A \circ A \subset A$. Let $a \in S$. Then, by Result 5.A, there exist $x, y \in S$ such that $a = xaya$. Thus

$$\begin{aligned} \mu_{A\circ A}(a) &= \bigvee_{a=pq}[\mu_A(p) \wedge \mu_A(q)] \\ &\geq \mu_A(xa) \wedge \mu_A(ya) && \text{(Since } a = xaya) \\ &\geq \mu_A(a) \wedge \mu_A(a) && \text{(Since } A \in \mathrm{IFLI}(S)) \\ &= \mu_A(a) \end{aligned}$$

and

$$\begin{aligned} \nu_{A\circ A}(a) &= \bigwedge_{a=pq}[\nu_A(p) \vee \nu_A(q)] \geq \nu_A(xa) \vee \nu_A(ya) \\ &\geq \nu_A(a) \vee \nu_A(a) = \nu_A(a). \end{aligned}$$

So $A \subset A \circ A$. Hence $A \circ A = A$.

($\Leftarrow$): Suppose the necessary condition holds. Let $L \in \mathrm{LI}(S)$ and let $a \in L$. By Result 2.B(2), $(\chi_L, \chi_{L^c}) \in \mathrm{IFLI}(S)$. Then, by the hypothesis,

$$(\chi_L, \chi_{L^c}) \circ (\chi_L, \chi_{L^c}) = (\chi_L, \chi_{L^c}).$$

Thus

$$\mu_{(\chi_L,\chi_{L^c})\circ(\chi_L,\chi_{L^c})}(a) = \chi_L(a) = 1$$

and

$$\nu_{(\chi_L,\chi_{L^c})\circ(\chi_L,\chi_{L^c})}(a) = \chi_{L^c}(a) = 0\ .$$

So $[(\chi_L, \chi_{L^c}) \circ (\chi_L, \chi_{L^c})](a) \neq (0, 1)$. Then

$$\bigvee_{a=pq}[\chi_L(p) \wedge \chi_L(q)] = 1 \text{ and } \bigwedge_{a=pq}[\chi_{L^c}(p) \vee \chi_{L^c}(q)] = 0.$$

Thus there exist $b, c \in S$ with $a = bc$ such that

$$\chi_L(b) = 1,\ \chi_{L^c}(b) = 0 \text{ and } \chi_L(c) = 1,\ \chi_{L^c}(c) = 0.$$

So $b \in L$, i.e., $a = bc \in LL$. Then $L \subset LL$. It is clear that $LL \subset L$. Thus $L = LL$. Hence S is left quasiregular. This complete the proof. ■

A semigroup S is said to be *intraregular* if for each $a \in S$, there exist $x, y \in S$ such that $a = xa^2y$.

Result 5.B[19, Theorem 6]. *Let S be a semigroup. Then S is both intraregular and left quasiregualr if and only if for each $B \in GBI(S)$, each $L \in LI(S)$ and each $R \in RI(S)$, $L \cap R \cap B \subset LRB$.*

We give a characterization of a semigroup that is both intraregular and left quasiregular in terms of intuitionistic fuzzy sets.

Theorem 5.3. *Let S be a semigroup. Then the following are equivalent:*

(1) S is both intraregular and left quasiregular.

(2) $B \cap C \cap A \subset B \circ C \circ A$ for each $A \in$ IFBI(S), each $B \in$ IFLI(S) and each $C \in$ IFRI(S).

(3) $B \cap C \cap A \subset B \circ C \circ A$ for each $A \in$ IFGBI(S), each $B \in$ IFLI(S) and each $C \in$ IFRI(S).

Proof. (2) $\Rightarrow$(3): It is clear.

(3) $\Rightarrow$(1): It can be seen as in the proof of Theorem 4.5[(5) implies (1)].

(1) $\Rightarrow$(2): Suppose the condition (1) holds. Let $A \in$ IFBI(S), let $B \in$ IFLI(S) and let $C \in$ IFRI(S). Let $a \in S$. Since S is left quasiregular, by Result 5.A, there exist $u, v \in S$ such that $a = uava$. Then

$$a = uava = u(xa^2y)va = ((ux)a)((a(yv))a).$$

Thus

$$\begin{aligned}
\mu_{B\circ C\circ A}(a) &= \bigvee_{a=pq}[\mu_B(p) \wedge \mu_{C\circ A}(q)] \\
&\geq \mu_B((ux)a) \wedge \mu_{C\circ A}((avy)a) \\
&\geq \mu_B(a) \wedge (\bigvee_{ayva=pq}[\mu_C \wedge \mu_A(q)]) && (\text{Since } B \in \text{IFLI}(S)) \\
&\geq \mu_B(a) \wedge \mu_C(a(yv)) \wedge \mu_A(a) \\
&\geq \mu_B(a) \wedge \mu_C(a) \wedge \mu_A(a) && (\text{Since } C \in \text{IFRI}(S)) \\
&= \mu_{B\cap C\cap A}(a)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{B\circ C\circ A}(a) &= \bigwedge_{a=pq}[\nu_B(p) \vee \nu_{C\circ A}(q)] \leq \nu_B((ux)a) \vee \nu_{C\circ A}((avy)a) \\
&\leq \nu_B(a) \vee (\bigwedge_{ayva=pq}[\nu_C \vee \nu_A(q)]) \leq \nu_B(a) \vee \nu_C(a(yv)) \vee \nu_A(a) \\
&\leq \nu_B(a) \vee \nu_C(a) \vee \nu_A(a) = \nu_{B\cap C\cap A}(a).
\end{aligned}$$

Hence $C \cap B \cap A \subset C \circ B \circ A$. This complete the proof. ■

References

[1] K.Atanassov, Intuitionistic fuzzy sets, Fuzzy Set and Systems, 20(1986)

87-96.

[2] B.Banerjee and D.Kr. Basnet, Intuitionistic fuzzy subrings and ideals, J.Fuzzy Math 11(1)(2003), 139-155.

[3] R.Biswas,Intuitionistic fuzzy subrings, Mathematical Forum x(1989), 37-46.

[4] J.Calais, Semi-groups quasi-inversifs, C.R.Accd. Sci.Paris 252(1961), 2357-2359.

[5] A.H.Clifford and G.B.Preston, The Algebraic Theory of Semigroups, Vol.1, AMS, Soc. Providence(1961).

[6] D.Çoker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Set and Systems, 88(1997) 81-89.

[7] D.Çoker and A.Haydar Es, On fuzzy compactness in intuitionistic fuzzy topological spaces, J.Fuzzy Math. 3(1995), 899-909.

[8] H.Gurcay, D.Çoker and A.Haydar Es, On fuzzy continuity in intuitionistic fuzzy topological spaces, J.Fuzzy Math. 5(1997), 365-378.

[9] K.Hur, S.Y.Jang and H.W.Kang, Intuitionistic fuzzy subgroupoids, International Journal of Fuzzy Logic and Intelligent Systems 3(1)(2003), 72-77.

[10] K.Hur, H.W.Kang and H.K.Song, Intuitionistic fuzzy subgroups and subrngs, Honam Math.J.25(1)(2003), 19-41.

[11] K.Hur, S.Y.Jang and H.W.Kang, Intuitionistic fuzzy subgroups and cosets, Honam Math.J.26(1)(2004), 17-41.

[12] K.Hur, Y.B.Jun and J.H.Ryou, Intuitionistic fuzzy topological groups, Honam Math.J.26(2)(2004), 163-192.

[13] K.Hur, J.H.Kim and J.H.Ryou, Intuitionistic fuzzy topological spaces, J.Korea Soc. Math. Edu. Ser. B : Pure Appl. Math. 11(3)(2004), 243-265.

[14] K.Hur, K.J.Kim and H.K.Song, Intuitionistic fuzzy ideals and bi-ideals, Honam Math. J.26(3)(2004), 309-330.

[15] K.Hur, S.Y.Jang and H.W.Kang, Intuitionistic fuzzy normal subgroups and intuitionistic fuzzy cosets, Honam Math. J. 26(4)(2004), 559-587.

[16]____________________, Intuitionistic fuzzy congruences on a lattice, J.Appl. Math. a Computing 18(2)(2005), 465-486.

[17] S.Lajos, A note on completely regular semigroups, Acta Sci. Math.(Szeged) (28)(1967), 261-265.

[18]____________, Theorems on (1,1)-ideals in semigroups II, Depart. of Mathematics, Karl Marx Univ. of Fconomics, Budapest(1975).

[19]____________, On generalized bi-ideals in semigroups,Coll. Math. Soc,Janos Bilyai, 20, Algebraic Theory of Semigroups,(G.Ponak,Ed.), North-Holland(1979), 335-340.

[20] S.J.Lee and E.P.Lee, The category of intuitionistic fuzzy topological spaces, Bull. Korean Math. Soc. 37(1)(2000), 63-76.

[21] Wang-jin Liu, Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and Systems 8(1982), 133-139.

[22] A.Rosenfeld, Fuzzy groups, J.Math.Anal.Appl. 35(1971), 512-517.

[23] L.A.Zadeh, Fuzzy sets, Inoform. and Centrol 8(1965), 338-353.

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