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On a Generalization of Bernstein - Chlodovsky Polynomials for Two Variables

Aydın Izgi

Ankara University, Faculty of Science,
Dept. of Mathematics, 06100 Tandogan, Ankara, Turkey
izgi@science.ankara.edu.tr

İbrahim Büyükyazıcı

Gazi University, Faculty of Education
Dep. of Mathematics Education, Kastamonu, Turkey
bibrahim@gazi.edu.tr

Abstract

In this note it is studied the following generalization of Bernstein-Chlodovsky polynomials

$$B_{n,m}(f;x,y)=\sum_{k=0}^{n}\sum_{j=0}^{m}f(\frac{k}{n}b_n,\frac{j}{m}b_m)\varphi_n^k(\frac{x}{b_n})\varphi_j^m(\frac{y}{b_m})$$

where $0\leq x\leq b_n,\ 0\leq y\leq b_m$ and (b_n) is the sequence of real numbers and increasing which satisfies $\lim\limits_{n\to\infty}b_n=\infty,\quad \lim\limits_{n\to\infty}\frac{b_n}{n}=0$ and , $\varphi_n^k(t)=\left(\begin{array}{c}n\\k\end{array}\right)t^k(1-t)^{n-k}$.

It may be also seen that $B_{n,m}(f;x,y)$ is linear positive operator [1], [2]. A theorem for convergence of $B_{n,m}(f;x,y)$ to $f(x,y)$ as $n,m\to\infty$

in the space of continuous function on semi axes satisfying $|f(x,\,y)|\leq C_f(1+x^2+y^2)$ is established. And it is discussed the rate of approximation. Also if f has continuous partial derivatives we give a theorem for approximating properties of this operator $B_{n,m}(f;x,y)$ and we give some examples.

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1 Introduction

The aim of this paper is to study the problem of the approximation of functions of two variables by means of Bernstein-Chlodovsky polynomials in a rectangular domain.

There are many investigations devoted to the problem of approximating continuous functions by classical Bernstein polynomials, as well as by two-dimensional Bernstein polynomials and their generalizations. We refer to papers [3], [6], [7].

On the other hand, Bernstein-Chlodovsky polynomials have not been studied so well and we don't know of papers, devoted to the two dimensional case. Some generalization of these polynomials in the one-dimensional case may be found in [4], [5].

In this paper, we will prove theorems on the approximation of continuous functions by the following Bernstein-Chlodovsky polynomials of two variables:

$$B_{n,m}(f;x,y)=\sum_{k=0}^{n}\sum_{j=0}^{m}f(\frac{k}{n}b_n,\frac{j}{m}b_m)\varphi_n^k(\frac{x}{b_n})\varphi_j^m(\frac{y}{b_m}) \tag{1}$$

where $0\leq x\leq b_n,\ 0\leq y\leq b_m$ and $\varphi_n^k(t)=\begin{pmatrix} n\\ k\end{pmatrix}t^k(1-t)^{n-k}$ and (b_n) is the sequence of real numbers and increasing which satisfies $\lim\limits_{n\to\infty}b_n=\infty$ and $\lim\limits_{n\to\infty}\frac{b_n}{n}=0$.

Also, it is investigated the theorem on convergence of polynomials in (1) and it is discussed in Theorem 2 the order of approximation of continuous functions by the sequence $B_{n,m}$. On the other hand if f has continuous partial derivatives, it is proved a new theorem for the approximating properties of Bernstein-Chlodovsky polynomials. It is given some numerical examples for the approximation.

2 Main Results: Convergence and Rate of Approximation

For any positive $A>0,\ B>0$ we denote the rectangular domain $[0,\,A]\times[0,\,B]$ by D_{AB} and by $D_{b_nb_m}$ the corresponding rectangular with $A=b_n$ and $B=b_m$.

Now, in this section we will be giving the theorem for the convergence. The domain $D_{b_nb_m}$ extends to the infinite quadrant $x\geq 0,\ y\geq 0$ as $n,m\to\infty$ and therefore we can establish some theorems on the convergence and the rate of approximation of continuous functions by polynomials (1) on unbounded set.

Theorem 2.1 *For any fixed positive real numbers $A > 0$, $B > 0$, the relation,*

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \max_{(x,y)\in D_{AB}} |B_{n,m}(f;\, x,\, y) - f(x,\, y)| = 0$$

holds for all functions f which are continuous in $x \geq 0$ and $y \geq 0$ and satisfy the condition

$$|f(x,\, y)| \leq C_f(1 + x^2 + y^2) \tag{2}$$

C_f is a constant depending on the function f only.

Proof: Firstly, we can easily prove the following Korovkin type equalities,

$$B_{n,m}(1; x, y) = 1 \tag{3}$$

$$B_{n,m}(t_1; x, y) = x \tag{4}$$

$$B_{n,m}(t_2; x, y) = y \tag{5}$$

$$B_{n,m}(t_1^2 + t_2^2; x, y) = x^2 + y^2 + \frac{x(b_n - x)}{n} + \frac{y(b_m - y)}{m} \tag{6}$$

If we use the above equalities we can see that,

$$\begin{aligned} \|B_{n,m}(1; x, y) - 1\|_{C(D_{AB})} &= 0,\ \|B_{n,m}(t_1; x, y) - x\|_{C(D_{AB})} = 0 \\ \text{and } \|B_{n,m}(t_2; x, y) - y\|_{C(D_{AB})} &= 0 \end{aligned}$$

$$\begin{aligned} \left\|B_{n,m}(t_1^2 + t_2^2; x, y) - (x^2 + y^2)\right\|_{C(D_{AB})} &= \max_{(x,\,y)\in D_{AB}} \left| \frac{x(b_n - x)}{n} + \frac{y(b_m - y)}{m} \right| \\ &\leq A\frac{b_n}{n} + B\frac{b_m}{m} \to 0 \end{aligned}$$

where $n \to \infty$, $m \to \infty$. We can apply Korovkin-type theorem for multivariate functions [8], The conditions in [8] remains true under the condition (2).

Given the region D_{AB} for some large n, m, $D_{b_n b_m}$ will contain D_{AB} and the theorem gives a solution of the approximation problem for closed subset of D_{AB} . Hence The proof of Theorem 1 is completed.

It is seen that for large n and m, the rectangular domain $D_{b_n b_m}$ coincides any region D_{AB} and therefore this theorem gives a solution of approximation problems of a continuous function, satisfying (2) only in any closed subset of $D_{b_n b_m}$.

Example 2.2 *For $(n\ , m) = \{(4, 5)\, , (8, 10)\, , (25, 15)\, , (30, 25)\}$, the convergence of $B_{n,m}(f; x, y)$ to $f(x, y) = \sqrt{xy}e^{(-(1/2)(x^2+y^2))}$ in the space of continuous function on semi axes satisfying $|f(x,\, y)| \leq C_f(1 + x^2 + y^2)$ is shown by following graphic.*

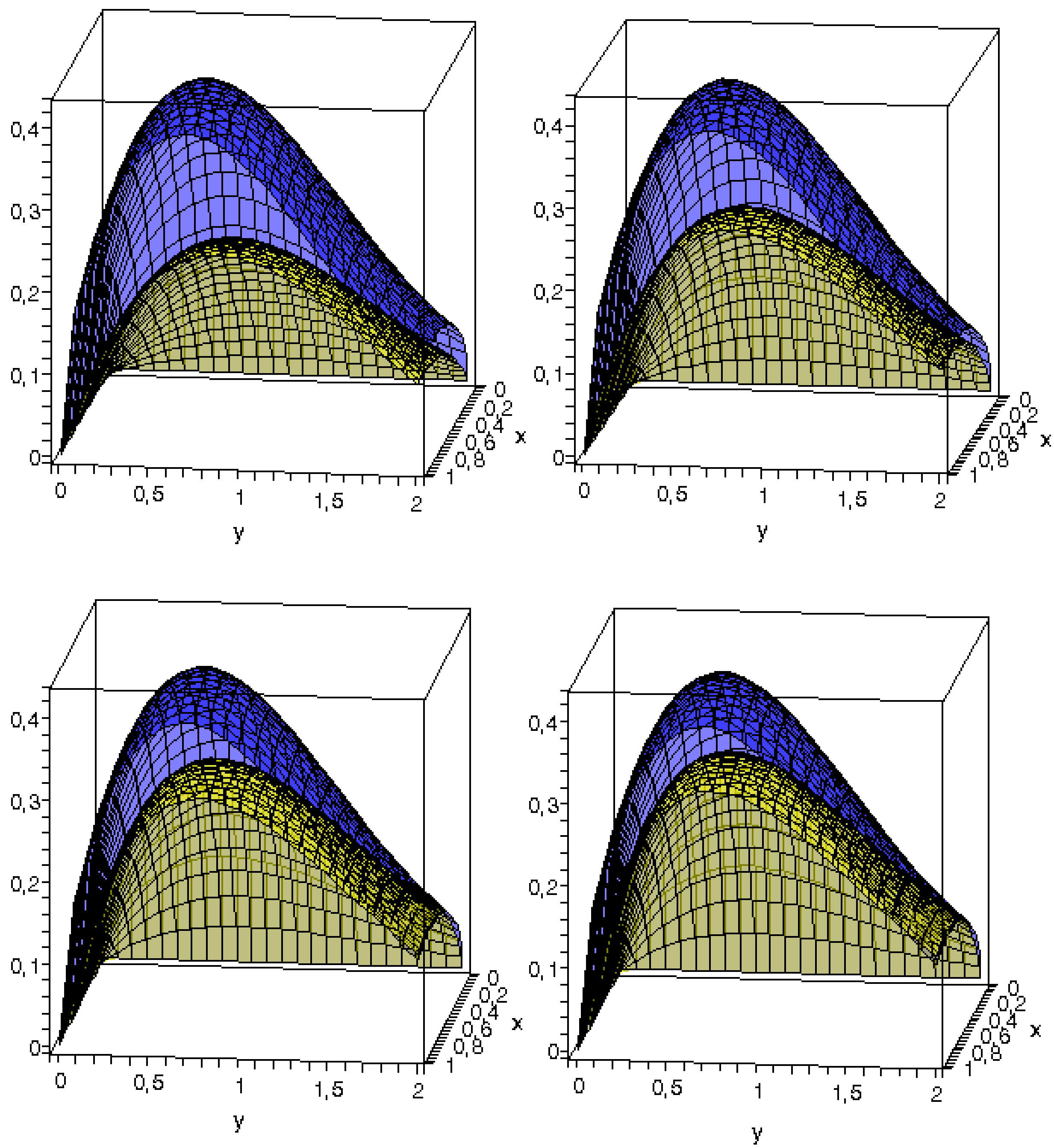

Figure 1. Graphics of approximation of $f \quad (x, y)$ for $n = 4, 8, 25, 30$ and $m = 5, 10, 15, 25$.

Now, in this part we will be giving the theorem for the rate of approximation. Also if f has continuous partial derivatives, we give another theorem for the rate of approximation.

Now we give modulus of continuity of function f on $x \geq 0, y \geq 0$. Let $x = (x_1,\, x_2),\ t = (t_1\ t_2),\ x^{(1)} = (x_1,\, u),\ x^{(2)} = (v,\, x_2),$
$t^{(1)} = (t_1,\, u),\ t^{(2)} = (v,\, t_2)$ and $\rho(x,\, t) = \sqrt{(t_1 - x_1)^2 + (t_2 - x_2)^2}$

$$\omega_{AB}(\delta) = \sup\{|f(t) - f(x)| : x, t \in D_{AB},\ \rho(x,\, t) \leq \delta\}$$

$$\omega_{AB}^{(1)}(\delta) = \sup\left\{\left|f(t^{(1)}) - f(x^{(1)})\right| \ :\ x^{(1)}, t^{(1)} \in D_{AB},\quad \rho(x^{(1)}, t^{(1)}) \leq \delta\right\}$$

$$\omega_{AB}^{(2)}(\delta) = \sup\left\{\left|f(t^{(2)}) - f(x^{(2)})\right| \ :\ x^{(2)}, t^{(2)} \in D_{AB},\quad \rho(x^{(2)}, t^{(2)}) \leq \delta\right\}$$

$\omega_{AB}(\delta)$ is the complete modulus of continuity of f, $\omega_{AB}^{(1)}(\delta)$ is the first partial modulus of continuity of f and $\omega_{AB}^{(2)}(\delta)$ is the second partial modulus of continuity of f.

The following theorem, it is established for rate of approximation. Here $\omega_{A'B'}$ denotes complete modulus of continuity of f where $A' = 1 + A,\ B' = 1 + B$.

Theorem 2.3 *Let f be continuous in $x \geq 0$ and $y \geq 0$ and satisfy the condition (2). Then for any fixed $A > 0$, $B > 0$,*

$$|B_{n,m}(f;\, x,\, y) - f(x,\, y)| \leq C.\omega_{A'B'}(\sqrt{\frac{b_n}{n} + \frac{b_m}{m}}) \tag{7}$$

holds. C is independent of $n,\ m$.

Proof. First we introduce four point sets. For $(x,\, y) \in D_{AB}$

$$E_1 = \{(k,\, j) : \frac{k}{n}b_n \geq A',\ \frac{j}{m}b_m \geq B'\},\quad E_2 = \{(k,\, j) : \frac{k}{n}b_n \leq A',\ \frac{j}{m}b_m \geq B'\}$$

$$E_3 = \{(k,\, j) : \frac{k}{n}b_n \geq A',\ \frac{j}{m}b_m \leq B'\},\quad E_4 = \{(k,\, j) : \frac{k}{n}b_n \leq A',\ \frac{j}{m}b_m \leq B'\},$$

then from (3) we obtain

$$\begin{aligned}
|B_{n,m}(f;\, x,\, y) - f(x,\, y)| \ \leq\ & \sum_{(k,j)\in}^{n}\sum_{E_1}^{m}\left|f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(x,y)\right|\Phi_{n,m}^{k,j}(\frac{x}{b_n}, \frac{y}{b_m}) \\
&+ \sum_{(k,j)\in}^{n}\sum_{E_2}^{m}\left|f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(x,y)\right|\Phi_{n,m}^{k,j}(\frac{x}{b_n}, \frac{y}{b_m}) \\
&+ \sum_{(k,j)\in}^{n}\sum_{E_3}^{m}\left|f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(x,y)\right|\Phi_{n,m}^{k,j}(\frac{x}{b_n}, \frac{y}{b_m}) \\
&+ \sum_{(k,j)\in}^{n}\sum_{E_4}^{m}\left|f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(x,y)\right|\Phi_{n,m}^{k,j}(\frac{x}{b_n}, \frac{y}{b_m})
\end{aligned}$$

$$= I_{n,m}^{(1)} + I_{n,m}^{(2)} + I_{n,m}^{(3)} + I_{n,m}^{(4)}$$

where $\Phi_{n,m}^{k,j}(\frac{x}{b_n}, \frac{y}{b_m}) = \varphi_n^k(\frac{x}{b_n})\varphi_j^m(\frac{y}{b_m})$. Consider $I_{n,m}^{(1)}$, Since $f \in C_O(R_0^2)$, using (2) we have

$$\begin{aligned} \left| f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(x,y) \right| &\leq C_f[2 + (\frac{k}{n}b_n)^2 + (\frac{j}{m}b_m)^2 + x^2 + y^2] \\ &\leq C_f[2 + (\frac{k}{n}b_n - x)^2 + (\frac{j}{m}b_m - y)^2] \\ &\quad +2C_f[x\left|\frac{k}{n}b_n - x\right| + y\left|\frac{j}{m}b_m - y\right| + x^2 + y^2]. \end{aligned}$$

Now it is clear that $\left|\frac{k}{n}b_n - x\right| \geq 1$, $\left|\frac{j}{m}b_m - y\right| \geq 1$ for $(k,j) \in E_1$ and $(x,y) \in D_{AB}$. By this reason we have

$$\left| f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(x,y) \right| \leq C_1[(\frac{k}{n}b_n - x)^2 + (\frac{j}{m}b_m - y)^2]$$

where $C_1 = 4C_f(2A + 2B + 1)^2$.

According to equalities (3)-(6)

$$I_{n,m}^{(1)} \leq \sum_{k=0}^{n}\sum_{j=0}^{m} \left| f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(x,y) \right| \Phi_{n,m}^{k,j}(\frac{x}{b_n}, \frac{y}{b_m})$$

$$\leq C_1[\frac{x(b_n - x)}{n} + \frac{y(b_m - y)}{m}] \leq C_2(\frac{b_n}{n} + \frac{b_m}{m})$$

where $C_2 = C_1(A + B)$.

By the same way, since $\left|\frac{j}{m}b_m - y\right| \geq 1$ for $(k,j) \in E_2$ and $(x,y) \in D_{AB}$, we have

$$I_{n,m}^{(2)} \leq C_3(\frac{b_n}{n} + \frac{b_m}{m})$$

and since $\left|\frac{k}{n}b_n - x\right| \geq 1$ for $(k,j) \in E_3$ and $(x,y) \in D_{AB}$, we have

$$I_{n,m}^{(3)} \leq C_4(\frac{b_n}{n} + \frac{b_m}{m}).$$

Thus if we choose $C_5 = C_2 + C_3 + C_4$ we obtain that

$$I_{n,m}^{(1)} + I_{n,m}^{(2)} + I_{n,m}^{(3)} \leq C_5(\frac{b_n}{n} + \frac{b_m}{m}).$$

Since $\frac{b_n}{n}+\frac{b_m}{m} \to 0$ $(n \to \infty, m \to \infty)$, we can write that $\frac{b_n}{n}+\frac{b_m}{m} \leq \sqrt{\frac{b_n}{n}+\frac{b_m}{m}}$ for n and m sufficiently large. Using properties of complete modulus of continuity, we have $C_6\sqrt{\frac{b_n}{n}+\frac{b_m}{m}} \leq \omega_{A'B'}(\sqrt{\frac{b_n}{n}+\frac{b_m}{m}})$.

Thus we obtain

$$I^{(1)}_{n,m}+I^{(2)}_{n,m}+I^{(3)}_{n,m} \leq C_7\omega_{A'B'}(\sqrt{\frac{b_n}{n}+\frac{b_m}{m}}) \tag{8}$$

where $C_7=\frac{C_5}{C_6}$.

Let $(k,j) \in E_4$ and $(x,y) \in D_{AB}$, then using the properties of modulus of continuity we obtain

$$\begin{aligned} \left|f(\frac{k}{n}b_n,\frac{j}{m}b_m)-f(x,y)\right| &\leq \omega_{A'B'}(\sqrt{(\frac{k}{n}b_n-x)^2+(\frac{j}{m}b_m-y)^2}) \\ &\leq \omega_{A'B'}(\delta_{n,m}).[1+\frac{1}{\delta_{n,m}}\sqrt{(\frac{k}{n}b_n-x)^2+(\frac{j}{m}b_m-y)^2}] \end{aligned}$$

where $\delta_{n,m}$ is a sequence which tends to zero as $n \to \infty$, $m \to \infty$.

According to equalities (3)-(6) and applying Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} I^{(4)}_{n,m} &\leq \omega_{A'B'}(\delta_{n,m}).[1+\frac{1}{\delta_{n,m}}\sqrt{\frac{x(b_n-x)}{n}+\frac{y(b_m-y)}{m}}] \\ &\leq \omega_{A'B'}(\delta_{n,m}).[1+\frac{1}{\delta_{n,m}}\sqrt{A+B}\sqrt{\frac{b_n}{n}+\frac{b_m}{m}}]. \end{aligned}$$

If we put $\delta_{n,m}=\sqrt{\frac{b_n}{n}+\frac{b_m}{m}}$, we have

$$I^{(4)}_{n,m} \leq [1+\sqrt{A+B}].\omega_{A'B'}(\sqrt{\frac{b_n}{n}+\frac{b_m}{m}}). \tag{9}$$

If we choose $C=C_7+[1+\sqrt{A+B}]$, by (8) and (9), the proof of the Theorem 2 is complete.

Example 2.4 *For $b_n=\sqrt{n}$ and $0 \leq x,y \leq 2$, in the following table, it can be seen that the approximation rate of function $f(x,y)=e^{-x}-xy+y^2$ for full continuity modulus of function f .*

(n,m)	$\lvert B_{n,m}(f;x,y)-f(x,y)\rvert \leq C\omega_{A',B'}(f;\sqrt{\frac{b_n}{n}+\frac{b_m}{m}})$
$(10^{15},10^{15})$	0.979407116
$(10^{15},10^{16})$	0.794536389
$(10^{16},10^{16})$	0.550761128
$(10^{16},10^{17})$	0.446800667
$(10^{17},10^{17})$	0.309715752
$(10^{17},10^{18})$	0.251254486
$(10^{18},10^{18})$	0.174165970
$(10^{18},10^{19})$	0.141290782
$(10^{19},10^{19})$	0.097940723
$(10^{19},10^{20})$	0.074536465
$(10^{20},10^{20})$	0.055076116
$(10^{20},10^{21})$	0.044680069
$(10^{21},10^{21})$	0.030971576
$(10^{21},10^{22})$	0.025125449
$(10^{22},10^{22})$	0.017416597
$(10^{22},10^{23})$	0.014129078
$(10^{23},10^{23})$	0.009794072
$(10^{23},10^{24})$	0.007945364
$(10^{24},10^{24})$	0.005507611
$(10^{24},10^{25})$	0.004468006
$(10^{25},10^{25})$	0.003097157

Table 1. Error of the order of the approximation of $f(x,y)=e^{-x}-xy+y^2$.

If the functions have continuous partial derivatives then we can give a new rate of approximation of f . Following theorem is related to this situation. In the following theorem $\varpi^{(1)}_{A'B'}$ is the first partial modulus of $\frac{\partial f}{\partial x}$ and $\varpi^{(2)}_{A'B'}$ is the second partial modulus of $\frac{\partial f}{\partial y}$, where $A'=1+A,$ and $B'=1+B$.

Theorem 2.5 *Let f be continuous in $x\geq 0$ and $y\geq 0$. If f has continuous partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ which satisfy*

$$\left|\frac{\partial f(x,\,y)}{\partial x}\right| \leq M_f(1+x+y), \qquad \left|\frac{\partial f(x,\,y)}{\partial y}\right| \leq N_f(1+x+y) \tag{10}$$

where M_f and N_f is dependent on f, then

$$|B_{n,m}(f;\, x,\, y) - f(x,\, y)| \leq M \left[\omega^{(2)}_{A'B'} \left(\sqrt{\frac{b_n}{n} + \frac{b_m}{m}} \right) + \sqrt{\frac{b_n}{n} + \frac{b_m}{m}} . \varpi^{(1)}_{A'B'} \left(\sqrt{\frac{b_n}{n} + \frac{b_m}{m}} \right) \right] \tag{11}$$

$$|B_{n,m}(f;\, x,\, y) - f(x,\, y)| \leq N \left[\omega^{(1)}_{A'B'} (\sqrt{\frac{b_n}{n} + \frac{b_m}{m}}) + \sqrt{\frac{b_n}{n} + \frac{b_m}{m}} . \varpi^{(2)}_{A'B'} (\sqrt{\frac{b_n}{n} + \frac{b_m}{m}}) \right] . \tag{12}$$

Proof: Due to mean value theorem we have

$$f(\frac{k}{n}b_n, y) - f(x, y) = (\frac{k}{n}b_n - x)\frac{\partial f(x,\, y)}{\partial x} + (\frac{k}{n}b_n - x)[\frac{\partial f(\zeta,\, y)}{\partial x} - \frac{\partial f(x,\, y)}{\partial x}] \tag{13}$$

for any fixed $y \in [0,\ B]$, where ζ is some point between x and $\frac{k}{n}b_n$. On the other hand we have

$$\begin{aligned} f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(x, y) \quad &= \quad f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(\frac{k}{n}b_n, y) \\ &\quad + f(\frac{k}{n}b_n, y) - f(x, y). \end{aligned}$$

Thus, using equality (3) for this equality we obtain

$$\begin{aligned} |B_{n,m}(f;\, x,\, y) - f(x,\, y)| \leq & \sum_{k=0}^{n} \sum_{j=0}^{m} \left| f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(\frac{k}{n}b_n, y) \right| \Phi^{k,j}_{n,m}(\frac{x}{b_n},\, \frac{y}{b_m}) \\ & + \sum_{k=0}^{n} \sum_{j=0}^{m} \left| f(\frac{k}{n}b_n, y) - f(x, y) \right| \Phi^{k,j}_{n,m}(\frac{x}{b_n},\, \frac{y}{b_m}) \\ & = q_{n,m}(x,\, y) + r_{n,m}(x,\, y). \end{aligned}$$

Considering the sets $E_i;\ i = 1,\, 2,\, 3,\, 4$ as in Theorem 2, we can write that

$$
\begin{aligned}
q_{n,m}(x,\,y) \;=\; & \sum_{(k,j)\in}^{n} \sum_{E_1}^{m} \left| f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(\frac{k}{n}b_n, y) \right| \Phi_{n,m}^{k,j}(\frac{x}{b_n},\,\frac{y}{b_m}) \\
& + \sum_{(k,j)\in}^{n} \sum_{E_2}^{m} \left| f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(\frac{k}{n}b_n, y) \right| \Phi_{n,m}^{k,j}(\frac{x}{b_n},\,\frac{y}{b_m}) \\
& + \sum_{(k,j)\in}^{n} \sum_{E_3}^{m} \left| f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(\frac{k}{n}b_n, y) \right| \Phi_{n,m}^{k,j}(\frac{x}{b_n},\,\frac{y}{b_m}) \\
& + \sum_{(k,j)\in}^{n} \sum_{E_4}^{m} \left| f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(\frac{k}{n}b_n, y) \right| \Phi_{n,m}^{k,j}(\frac{x}{b_n},\,\frac{y}{b_m}) \\
=\; & q_{n,m}^{(1)} + q_{n,m}^{(2)} + q_{n,m}^{(3)} + q_{n,m}^{(4)}
\end{aligned}
$$

since f is continuous, using (2) we have

$$
\begin{aligned}
\left| f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(\frac{k}{n}b_n, y) \right| &\le C_f \left[2 + 2\left(\frac{k}{n}b_n\right)^2 + \left(\frac{j}{m}b_m\right)^2 + y^2 \right] \\
&\le C_f \left[2 + 2\left(\frac{k}{n}b_n - x\right)^2 + \left(\frac{j}{m}b_m - y\right)^2 \right] \\
&+ C_f \left[4x\left|\frac{k}{n}b_n - x\right| + 2y\left|\frac{j}{m}b_m - y\right| + 2(x^2+y^2) \right].
\end{aligned}
$$

Consider $q_{n,m}^{(1)}$; Now it is clear that

$$
\left|\frac{k}{n}b_n - x\right| \ge 1\,, \quad \left|\frac{j}{m}b_m - y\right| \ge 1
$$

for $(k,j)\in E_1$ and $(x,y)\in D_{AB}$. By this reason we have

$$
\left| f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(\frac{k}{n}b_n, y) \right| \le C_8 \left[\left(\frac{k}{n}b_n - x\right)^2 + \left(\frac{j}{m}b_m - y\right)^2 \right]
$$

where $C_8 = 2C_f(A+B+2)^2$. According to equalities (3)-(6),

$$
\begin{aligned}
q_{n,m}^{(1)} &\le C_8. \sum_{k=0}^{n}\sum_{j=0}^{m} \left[\left(\frac{k}{n}b_n - x\right)^2 + \left(\frac{j}{m}b_m - y\right)^2 \right] .\Phi_{n,m}^{k,j}(\frac{x}{b_n},\,\frac{y}{b_m}) \\
&\le C_8. \left(\frac{x(b_n-x)}{n} + \frac{y(b_m-y)}{m} \right)
\end{aligned}
$$

$$q_{n,m}^{(1)} \leq C_9.(\frac{b_n}{n} + \frac{b_m}{m})$$

where $C_9 = C_8.(A + B)$.

By the same way, since $\left|\frac{j}{m}b_m - y\right| \geq 1$ for $(k,j) \in E_2$ and $(x,y) \in D_{AB}$, we have

$$q_{n,m}^{(2)} \leq C_{10}(\frac{b_n}{n} + \frac{b_m}{m})$$

and since $\left|\frac{k}{n}b_n - x\right| \geq 1$ for $(k,j) \in E_3$ and $(x,y) \in D_{AB}$, we have

$$q_{n,m}^{(3)} \leq C_{11}(\frac{b_n}{n} + \frac{b_m}{m}).$$

Thus if we chose $C_{12} = C_9 + C_{10} + C_{11}$ we obtain that

$$q_{n,m}^{(1)} + q_{n,m}^{(2)} + q_{n,m}^{(3)} \leq C_{12}(\frac{b_n}{n} + \frac{b_m}{m}).$$

Since $\frac{b_n}{n} + \frac{b_m}{m} \to 0$ $(n \to \infty, m \to \infty)$, we can write that; $\frac{b_n}{n} + \frac{b_m}{m} \leq \sqrt{\frac{b_n}{n} + \frac{b_m}{m}}$ for n and m sufficiently large. Using properties of modulus of continuity, we have
$C_{13}\sqrt{\frac{b_n}{n} + \frac{b_m}{m}} \leq \omega_{A'B'}^{(2)}(\sqrt{\frac{b_n}{n} + \frac{b_m}{m}})$. Thus we obtain

$$q_{n,m}^{(1)} + q_{n,m}^{(2)} + q_{n,m}^{(3)} \leq C_{14}\omega_{A'B'}^{(2)}(\sqrt{\frac{b_n}{n} + \frac{b_m}{m}})$$

where $C_{14} = \frac{C_{12}}{C_{13}}$.

Let $(k,j) \in E_4$ and $(x,y) \in D_{AB}$, then using properties of modulus of continuity we obtain

$$\begin{aligned}\left|f(\frac{k}{n}b_n, \frac{j}{m}b_m) - f(\frac{k}{n}b_n, y)\right| &\leq \omega_{A'B'}^{(2)}(\left|\frac{j}{m}b_m - y\right|) \\ &\leq \omega_{A'B'}^{(2)}(\delta_{n,m})[1 + \frac{\left|\frac{j}{m}b_m - y\right|}{\delta_{n,m}}]\end{aligned}$$

According to equalities (3)-(6) and applying Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} q_{n,m}^{(4)} &\leq \omega_{A'B'}^{(2)}(\delta_{n,m}) \left[1+\frac{1}{\delta_{n,m}}\sqrt{\sum_{j=0}^{m}(\frac{j}{m}b_m-y)^2\varphi_m^j(\frac{y}{b_m})}\right] \\ &\leq \omega_{A'B'}^{(2)}(\delta_{n,m}) \left[1+\frac{1}{\delta_{n,m}}\sqrt{\frac{y(b_m-y)}{m}}\right] \\ &\leq \omega_{A'B'}^{(2)}(\delta_{n,m}) \left[1+\frac{1}{\delta_{n,m}}\sqrt{B}\sqrt{\frac{b_n}{n}+\frac{b_m}{m}}\right]. \end{aligned}$$

If we put $\delta_{n,m}=\sqrt{\frac{b_n}{n}+\frac{b_m}{m}}$, we have

$$q_{n,m}^{(4)} \leq C_{15}\omega_{A'B'}^{(2)}(\sqrt{\frac{b_n}{n}+\frac{b_m}{m}})$$

where $C_{15}=(1+\sqrt{B})$. Thus,

$$q_{n,m}(x,\,y) \leq C_{16}\omega_{A'B'}^{(2)}(\sqrt{\frac{b_n}{n}+\frac{b_m}{m}}) \tag{14}$$

where $C_{16}=C_{14}+C_{15}$, $(x,\,y)\in D_{AB}$.

Now we introduce two point sets for $(x,\,y)\in D_{AB}$.

$$K_1=\{(k,\,j):\frac{k}{n}b_n\geq A'\}, \qquad K_2=\{(k,\,j):\frac{k}{n}b_n\leq A'\}.$$

Using (13)

$$\begin{aligned} r_{n,m}(x,\,y) &= \left|\sum_{k=0}^{n}\sum_{j=0}^{m}(f(\frac{k}{n}b_n,y)-f(x,y))\Phi_{n,m}^{k,j}(\frac{x}{b_n},\,\frac{y}{b_m})\right| \\ &\leq \left|\frac{\partial f(x,\,y)}{\partial x}\sum_{k=0}^{n}\sum_{j=0}^{m}(\frac{k}{n}b_n-x)\Phi_{n,m}^{k,j}(\frac{x}{b_n},\,\frac{y}{b_m})\right| \\ &\quad +\left|\sum_{k=0}^{n}\sum_{j=0}^{m}(\frac{k}{n}b_n-x)[\frac{\partial f(\zeta,\,y)}{\partial x}-\frac{\partial f(x,\,y)}{\partial x}]\Phi_{n,m}^{k,j}(\frac{x}{b_n},\,\frac{y}{b_m})\right|. \end{aligned}$$

The first term at the last expression is zero by (4). Thus

$$r_{n,m}(x,\,y) \leq \sum_{k=0}^{n}\sum_{j=0}^{m}\left|\frac{k}{n}b_n-x\right|\left|\frac{\partial f(\zeta,\,y)}{\partial x}-\frac{\partial f(x,\,y)}{\partial x}\right|\Phi_{n,m}^{k,j}(\frac{x}{b_n},\,\frac{y}{b_m})$$

$$=(\sum_{(k,\,j)}^{n}\sum_{\in K_1}^{m}+\sum_{(k,\,j)}^{n}\sum_{\in K_2}^{m})\left|\frac{k}{n}b_n-x\right|\left|\frac{\partial f(\zeta,\,y)}{\partial x}-\frac{\partial f(x,\,y)}{\partial x}\right|\Phi_{n,m}^{k,j}(\frac{x}{b_n},\,\frac{y}{b_m})$$

$$= r^{(1)}_{n,m} + r^{(2)}_{n,m}.$$

From (10) and since $|\zeta - x| \leq \left|\frac{k}{n}b_n - x\right|$, we have,

$$\begin{aligned}\left|\frac{\partial f(\zeta, y)}{\partial x} - \frac{\partial f(x, y)}{\partial x}\right| &\leq M_f[2 + \zeta + x + 2y] \\ &\leq M_f[|\zeta - x| + 2(1 + x + y)] \\ &\leq M_f[\left|\frac{k}{n}b_n - x\right| + 2(1 + x + y)]\end{aligned}$$

Furthermore, if we consider $\left|\frac{k}{n}b_n - x\right| \geq 1$ for $(k, j) \in K_1$ and $(x, y) \in D_{AB}$.

$$\left|\frac{\partial f(\zeta, y)}{\partial x} - \frac{\partial f(x, y)}{\partial x}\right| \leq C_{17}\left|\frac{k}{n}b_n - x\right|$$

where $C_{17} = M_f[3 + 2(A + B)]$.

Using equalities (3)-(6) and the properties of modulus of continuity we obtain

$$\begin{aligned}r^{(1)}_{n,m} &\leq C_{17}\sum_{k=0}^{n}\sum_{j=0}^{m}(\frac{k}{n}b_n - x)^2\Phi^{k,j}_{n,m}(\frac{x}{b_n}, \frac{y}{b_m}) \\ &= C_{17}\frac{x(b_n - x)}{n} \leq C_{17}A(\frac{b_n}{n} + \frac{b_m}{m}) \\ &= C_{17}A\sqrt{\frac{b_n}{n} + \frac{b_m}{m}}.\sqrt{\frac{b_n}{n} + \frac{b_m}{m}} \\ &\leq C_{18}\sqrt{\frac{b_n}{n} + \frac{b_m}{m}}\varpi^{(1)}_{A'B'}(\sqrt{\frac{b_n}{n} + \frac{b_m}{m}})\end{aligned}$$

where $C_{18} = \frac{C_{17}A}{C_f}$, C_f dependent f and $\sqrt{\frac{b_n}{n} + \frac{b_m}{m}}$.

Let $(k, j) \in K_2$ and $(x, y) \in D_{AB}$. Thus $\left|\frac{k}{n}b_n - x\right| \leq 1$ and if we consider $|\zeta - x| \leq \left|\frac{k}{n}b_n - x\right|$ we have

$$\begin{aligned}\left|\frac{\partial f(\zeta, y)}{\partial x} - \frac{\partial f(x, y)}{\partial x}\right| &\leq \varpi^{(1)}_{A'B'}(|\zeta - x|) \\ &\leq \varpi^{(1)}_{A'B'}(\left|\frac{k}{n}b_n - x\right|) \\ &\leq \varpi^{(1)}_{A'B'}(\delta_{n,m})[1 + \frac{1}{\delta_{n,m}}\left|\frac{k}{n}b_n - x\right|].\end{aligned}$$

By the using equalities (3)-(6) and inequality of Cauchy-Schwartz,

$$\begin{aligned} r_{n,m}^{(2)} &\leq \varpi_{A'B'}^{(1)}(\delta_{n,m}) \sum_{k=0}^{n} \sum_{j=0}^{m} [\left| \frac{k}{n} b_n - x \right| + \frac{1}{\delta_{n,m}} (\frac{k}{n} b_n - x)^2] \Phi_{n,m}^{k,j}(\frac{x}{b_n}, \frac{y}{b_m}) \\ &\leq \varpi_{A'B'}^{(1)}(\delta_{n,m})[\sqrt{A}\sqrt{\frac{b_n}{n} + \frac{b_m}{m}} + A(\frac{b_n}{n} + \frac{b_m}{m})] \\ &\leq \sqrt{\frac{b_n}{n} + \frac{b_m}{m}}.\varpi_{A'B'}^{(1)}(\delta_{n,m})[\sqrt{A} + \frac{1}{\delta_{n,m}} A \sqrt{\frac{b_n}{n} + \frac{b_m}{m}}]. \end{aligned}$$

If we put $\delta_{n,m} = \sqrt{\frac{b_n}{n} + \frac{b_m}{m}}$ we obtain

$$r_{n,m}^{(2)} \leq [\sqrt{A} + A]\sqrt{\frac{b_n}{n} + \frac{b_m}{m}}.\varpi_{A'B'}^{(1)}(\sqrt{\frac{b_n}{n} + \frac{b_m}{m}}).$$

Set $C_{19} = C_{18} + [\sqrt{A} + A]$

$$r_{n,m}(x, y) \leq C_{19}\sqrt{\frac{b_n}{n} + \frac{b_m}{m}}.\varpi_{A'B'}^{(1)}(\sqrt{\frac{b_n}{n} + \frac{b_m}{m}}). \tag{15}$$

Finally, by (14) and (15)

$$|B_{n,m}(f; x, y) - f(x, y)|$$

$$\leq M[\omega_{A'B'}^{(2)}(\sqrt{\frac{b_n}{n} + \frac{b_m}{m}}) + \sqrt{\frac{b_n}{n} + \frac{b_m}{m}}.\varpi_{A'B'}^{(1)}(\sqrt{\frac{b_n}{n} + \frac{b_m}{m}}]$$

where $M = \max\{C_{16},\ C_{19}\}$.

We can prove (12) by the same way proof of (11). Hence Theorem 3 is proved.

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