International Mathematical Forum, Vol. 6, 2011, no. 51, 2535 - 2538

Existence Results for Fractional Fuzzy Differential Equations with Finite Delay

Haihua Wang and Yanqiong Liu

Department of Mathematics
Hunan University of Science and Technology
Xiangtan, Hunan, 411201, P.R. China
yql_math@yahoo.com.cn

Abstract

In this paper, we consider a class of boundary value problem for fuzzy differential equations of fractional order with finite delay. The existence of solution is established by the contraction mapping principle.

Mathematics Subject Classification: 34A12, 26E50, 45G10

Keywords: Fuzzy differential equation; Fractional order; Fuzzy solution; Existence

1 Introduction

The purpose of this paper is to investigate the existence of fuzzy solution for three-point boundary value problem for fractional differential equation:

$$\begin{cases} D^\alpha x(t) = f(t, x_t, D^\beta x(t)), \quad t \in J := [0,1],\ 1 < \alpha < 2, \\ x(t) = \phi(t), \quad t \in [-r, 0], \\ x(\xi) = x(1), \end{cases} \tag{1.1}$$

where D^α, D^β are standard Riemann-Liouville fractional derivatives, $\alpha - \beta \geq 1$, $\xi \in [0,1)$, $f : J \times C_0 \times E^n \to E^n$ is a fuzzy function, $\phi \in C_0$, $\phi(0) = \hat{0}$ and $C_0 = C([-r, 0], E^n)$. For any function x defined on $[-r, 1]$ and any $t \in J$, we denote by x_t the element of C_0 defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$.

To the best of our knowledge, although various results for fuzzy differential equations have been established until now, results for fuzzy differential equations involving fractional order are rarely seen, papers [1, 2] related to it only. The aim of the present paper is to establish some simple criteria for the existence and uniqueness of solution of the probelm(1.1). The paper is organized as follows. In Section 2, we present some preliminaries and lemmas. In Section 3, we discuss the existence of solution for probelm(1.1).

2 Preliminaries and Lemmas

Let $P_k(\mathbb{R}^n)$ be the family of all nonempty compact convex subsets of $\mathbb{R}^n$. For $A, B \in P_k(\mathbb{R}^n)$, the Hausdorff metric is defined by

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}.$$

A fuzzy set in $\mathbb{R}^n$ is a function with domain $\mathbb{R}^n$ and values in $[0,1]$, that is, an element of $[0,1]^{\mathbb{R}^n}$.

Let $u \in [0,1]^{\mathbb{R}^n}$, the α-level set is

$$[u]^\alpha = \{x \in \mathbb{R}^n | u(x) \geq \alpha\}, \quad \alpha \in (0,1].$$

By E^n, we denote the family of all upper-semicontinuous, normal fuzzy convex sets with $[u]^0 = \mathrm{cl}\ \{x \in \mathbb{R}^n | u(x) > 0\}$ is compact.

Let $d : E^n \times E^n \to [0, +\infty)$ be defined by

$$d(u,v) = \sup\{d_H([u]^\alpha, [v]^\alpha) | \alpha \in [0,1]\},$$

Then, (E^n, d) is a complete metric space. We define $\hat{0} \in E^n$ as $\hat{0}(0) = 1$ if $x = 0$ and $\hat{0} = 0$ if $x \neq 0$.

Definition 2.1 *The Riemann-Liouville fractional integral of order α for a function $f : [a,b] \to E^n$ is defined by*

$$I_{a+}^\alpha f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \quad \alpha > 0.$$

When $a = 0$, we write $I^\alpha f(t)$.

Definition 2.2 *For a function $f : [a,b] \to E^n$, the Riemann-Liouville derivative of fractional order $\alpha > 0$ is defined by*

$$D_{a+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)ds, \quad n = [\alpha] + 1.$$

Denote by $C(J, E^n)$ the set of all continuous mappings from J to E^n, the metric on $C(J, E^n)$ is defined by $H(u,v) = \sup_{t\in J} d(u(t), v(t))$. And we metricize C_0 by setting

$$H_0(x,y) = \max\{d(x(t), y(t)) | t \in [-r, 0]\}$$

for all $x, y \in C_0$. Set $X = \{x | x \in C([-r,1], E^n), D^\beta x \in C(J, E^n),$ and $x(t) = \phi(t), t \in [-r, 0]\}$, the metric on X will be defined later.

3 Main Result

Theorem 3.1 *Assume that* $f : J \times D_r \times E^n \to E^n$ *and there exist positive constants* K, L *such that*

$$d(f(t,u,D^\beta u), f(t,v,D^\beta v)) \leq KH_0(u,v) + Ld(D^\beta u, D^\beta v),$$

for all $t \in J$ *and all* $u, v \in C_0$. *Then*

$$\left(\frac{K}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha-\beta+1)}\right)\left(1 + \frac{1+\xi^\alpha}{1-\xi^{\alpha-1}}\right) < 1,$$

implies that the problem (1.1) *has a unique fuzzy solution on* $[-r, 1]$.

Proof. The metric H on X is defined by $H(u,v) = K \max_{t\in[-r,1]} d(u(t), v(t)) + L \max_{t\in[0,1]} d(D^\beta u, D^\beta v)$, $u, v \in X$. Then (X, H) is a complete metric space.

Transform the problem into a fixed point problem. It is clear that the solutions of problem (1.1) are fixed points of the operator $F : X \to X$ defined by

$$F(x)(t) = \begin{cases} \phi(t), \quad t \in [-r, 0], \\ \frac{\int_0^t (t-s)^{\alpha-1} f(s,x_s,D^\beta x(s))ds}{\Gamma(\alpha)} + \frac{\int_0^\xi (\xi-s)^{\alpha-1} f(s,x_s,D^\beta x(s))ds}{\Gamma(\alpha)(1-\xi^{\alpha-1})} t^{\alpha-1} \\ \quad - \frac{\int_0^1 (1-s)^{\alpha-1} f(s,x_s,D^\beta x(s))ds}{\Gamma(\alpha)(1-\xi^{\alpha-1})} t^{\alpha-1}, \quad t \in [0,1], \end{cases} \tag{3.1}$$

For $u, v \in X$, then

$$d(Fu(t), Tv(t)) = 0, \quad t \in [-r, 0], \tag{3.2}$$

and for $t \in J$, we have

$$\begin{aligned} & d(Fu(t), Fv(t)) \\ & \leq \frac{\int_0^t (t-s)^{\alpha-1} d(f(s,u_s,D^\beta u(s)), f(s,v_s,D^\beta v(s)))ds}{\Gamma(\alpha)} \\ & \quad + \frac{t^{\alpha-1}\int_0^\xi (\xi-s)^{\alpha-1} d(f(s,u_s,D^\beta u(s)), f(s,v_s,D^\beta v(s)))ds}{\Gamma(\alpha)(1-\xi^{\alpha-1})} \\ & \quad + \frac{t^{\alpha-1}\int_0^1 (1-s)^{\alpha-1} d(f(s,u_s,D^\beta u(s)), f(s,v_s,D^\beta v(s)))ds}{\Gamma(\alpha)(1-\xi^{\alpha-1})} \\ & \leq \frac{1}{\Gamma(\alpha)}\Bigg(\int_0^t (t-s)^{\alpha-1}[Kd(u_s(\theta), v_s(\theta)) + Ld(D^\beta u(s), D^\beta v(s))]ds \\ & \quad + \frac{t^{\alpha-1}}{(1-\xi^{\alpha-1})}\int_0^\xi (\xi-s)^{\alpha-1}[Kd(u_s(\theta), v_s(\theta)) + Ld(D^\beta u(s), D^\beta v(s))]ds \\ & \quad + \frac{t^{\alpha-1}}{(1-\xi^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1}[Kd(u_s(\theta), v_s(\theta)) + Ld(D^\beta u(s), D^\beta v(s))]ds\Bigg) \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\bigg(\int_0^t (t-s)^{\alpha-1}ds + \frac{t^{\alpha-1}}{(1-\xi^{\alpha-1})}\int_0^\xi (\xi-s)^{\alpha-1}ds \\
&\quad + \frac{t^{\alpha-1}}{(1-\xi^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1}ds\bigg) H_0(u,v) \qquad (3.3)\\
&\leq \frac{1}{\Gamma(\alpha+1)}\sup_{t\in J}\left(t^\alpha + \frac{\xi^\alpha t^{\alpha-1}}{1-\xi^{\alpha-1}} + \frac{t^{\alpha-1}}{1-\xi^{\alpha-1}}\right) H(u,v),
\end{aligned}
$$

and, similarly

$$
\begin{aligned}
&d(D^\beta Fu(t), D^\beta Fv(t)) \\
&\leq \frac{\int_0^t (t-s)^{\alpha-\beta-1} d(f(s,u_s,D^\beta u(s)), f(s,v_s,D^\beta v(s)))ds}{\Gamma(\alpha-\beta)} \\
&\quad + \frac{t^{\alpha-\beta-1}\int_0^\xi (\xi-s)^{\alpha-1} d(f(s,u_s,D^\beta u(s)), f(s,v_s,D^\beta v(s)))ds}{(1-\xi^{\alpha-1})} \qquad (3.4)\\
&\quad + \frac{t^{\alpha-\beta-1}\int_0^1 (1-s)^{\alpha-1} d(f(s,u_s,D^\beta u(s)), f(s,v_s,D^\beta v(s)))ds}{(1-\xi^{\alpha-1})}\bigg) \\
&\leq \frac{1}{\Gamma(\alpha-\beta+1)}\sup_{t\in J}\left(t^{\alpha-\beta} + \frac{\xi^\alpha t^{\alpha-\beta-1}}{1-\xi^{\alpha-1}} + \frac{t^{\alpha-\beta-1}}{1-\xi^{\alpha-1}}\right) H(u,v).
\end{aligned}
$$

Therefore, for each $t \in J$, we have

$$
H(Fu, Fv) \leq \left(\frac{K}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha-\beta+1)}\right)\left(1 + \frac{1+\xi^\alpha}{1-\xi^{\alpha-1}}\right) H(u,v).
$$

So, F is a contraction and thus F has a unique fixed point x on X, then $x(t)$ is the unique solution to problem (1.1) on $[-r, 1]$. □

ACKNOWLEDGEMENTS. The authors are supported financially by the Foundation of Hunan University of Science and Technology (E51054).

References

[1] R.P. Agarwal, V. Lakshmikantham, J.J. Nieto, *On the concept of solution for fractional differential equations with uncertainty,* Nonlinear Anal. **72**(2010):2859-2862.

[2] J.U. Jeong, *Existence results for fractional order fuzzy differential equations with infinite delay,* Int. Math. Forum. **5**(2010):3221-3230.

Received: March, 2011